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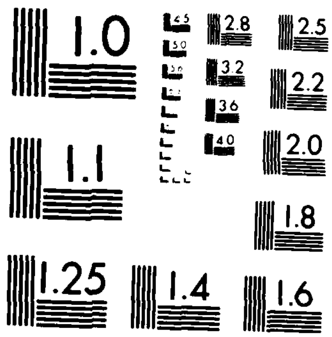


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# ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTION OF TEST STATISTICS ASSOCIATED WITH SEVERAL TWO PARAMETER EXPONENTIAL DISTRIBUTIONS

THESIS

David J. Lawton
Captain, USAF







DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY
AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

AFIT/GM/MATH/84D-1

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THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

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Master of Science in Applied Mathematics

David J. Lawton, B.S.

Captain, USAF

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## Preface

Many events occuring in nature have an underlying exponential distribution. This thesis obtained the distribution of three test criteria and tables of their percentage points are computed. These tables allow us to determine whether p exponential populations are equivalent, where $p > 2$. Methods were previously available to allow testing for equivalency of two populations, the test statistics presented in this paper and their distributions empower the testing of equality of more than two populations. I believe that anyone desiring to determine if up to 10 populations have the same exponential distribution will find this a valuable tool.

I am deeply indebted to my advisor Dr. B. N. Nagarsenker for his guidance and acknowledge his instruction on the theory and techniques on which this thesis is based. I also thank my readers Dr. P. B. Nagarsenker and Lt Col Thomas Lanier for their constructive remarks. The superb work of Susan Beachy, my typist reflects her patience and diligence in typing this project.

I formally thank my wife, Susan, for encouragement, forbearance, and enthusiasm to see me through this project and the AFIT program. The loving playfulness of our son Daniel has been a continuous source of refreshment and perspective. I dedicate this thesis to Jesus Christ, our faithful Saviour whose constant strength and matchless wisdom have sustained me during this endeavor.

David J. Lawton

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## Notation

| | |
|---|---|
| c. d. f. | :Cumulative distribution function |
| Def | :Definition |
| log | :Logarithm to the base e |
| LRT | :Likelihood-ratio test |
| M. G. F. | :Moment generating function |
| p. d. f. | :Probability density function |
| $\sim$ | :Is distributed |

## Abstract

This thesis uses three criteria to test for equality of p populations with underlying two parameter exponential distributions ($\theta$=location parameter, $\sigma$=scale parameter). These criteria use n random samples drawn from each of the p populations. The three criteria are based on the following three hypotheses:

1. $H_0 : \theta_1 = \theta_2 = \ldots = \theta_p$,

   $\sigma_1 = \sigma_2 = \ldots = \sigma_p$

2. $H_1 : \sigma_1 = \sigma_2 = \ldots = \sigma_p$

   $\theta_i$'s are unspecified $(i=1,2,\ldots,p)$

3. $H_2 : \theta_1 = \theta_2 = \ldots = \theta_p$

   given that $\sigma_1 = \sigma_2 = \ldots = \sigma_p$

against the general alternatives.

The asymptotic expansions of the distributions for $\lambda_1$, $\lambda_2$, and $\lambda_2$ are found based on the Neyman-Pearson likelihood ratio, where $\lambda_o$, $\lambda_1$, and $\lambda_2$ are the criteria for $H_o$, $H_1$, and $H_2$ respectively. The asymptotic expansions are computed using Bernoulli polynomials and a recursive relationship developed by Kalinin and Shalaevskii. Nine tables of percentage points are computed for each test statistic from the expansions where p = 2(1)10, n = 10(1)20(5)50(10)100, and $\alpha$ = .100, 050, .025, .010, .005. These tables along with a practical illustration give the analyst a good technique that can be applied to many exponentially related situations.

ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTION OF TEST STATISTICS ASSOCIATED WITH SEVERAL TWO PARAMETER EXPONENTIAL DISTRIBUTIONS

## I. Introduction

Statistical techniques allow analysts to examine the characteristics of events and determine if these events can be transformed into a known model. One model useful in describing the observed variation in events occurring "randomly in time" is the exponential distribution. Some of the situations in which the exponential distribution is appropriate are the interval between successive telephone calls, the interval between consecutive accidents incurred by the same individual worker, the "time to failure" of identical electronic components, and the time interval between industrial accidents.

This paper will develop tests, their distributions, and tables of percentage points to test if several two parameter exponential populations are the same on the basis of samples of size 'n' taken from these populations using methods analogous to analysis of variance.

Thus suppose that p samples are available and that the $i^{th}$ sample contains n observations $X_{ij}$, with mean $\overline{X}_i$ $(i=1,2,\ldots,p;j=1,2,\ldots,n)$ and have been drawn from an exponential distribution with probability density given by

$$p(x,\theta_i,\sigma_i)=\begin{cases}\dfrac{1}{\sigma_i}\exp\left[\dfrac{-(x-\theta_i)}{\sigma_i}\right] & x>\theta_i\ \ \sigma_i>o\ \ \theta_i>o \\ o & \text{otherwise} \\ & (i=1,2,\ldots,p)\end{cases} \tag{1.1}$$

where $\theta_i$ is the location parameter or the distance from the origin to the start of the distribution curve and $\sigma_i$ is the scale parameter of the exponential distribution.

The following hypotheses and the corresponding test procedures using the likelihood ratio criteria will be considered in this thesis.

<u>Null Hypothesis $H_o$</u>: The test of hypothesis $H_o$ that the p samples have been randomly drawn from the same population is equivalent to testing that the p exponential populations in equation 1.1 are identi·.l. In other words it is desired to test the hypothesis $H_o$:

$$\theta_1=\theta_2=\ldots=\theta_p \tag{1.2}$$

and $$\sigma_1=\sigma_2=\ldots=\sigma_p \tag{1.3}$$

against general alternatives.

<u>Null Hypothesis $H_1$</u>: The second null hypothesis $H_1$ tests to determine if the p scale parameters ($\sigma_i$) are identical, but with the location parameters ($\theta_i$) being any value whatsoever or equation 1.3 is true, while $\theta_i$'s take on any combination of values.

<u>Null Hypothesis $H_2$</u>: The null hypothesis $H_2$ tests to determine if the p populations have the same location parameter given they have the same scale parameter, or tests to determine if equation 1.2 is true given that equation 1.3 is true.

Because of the difficulties of obtaining the exact distributions of the likelihood ratio criteria for tests of hypothesis about the location and scale parameters of more than two exponential distributions, this thesis obtains an asymptotic expansion of the distributions of the three test statistics up to order $O(n^{-4})$. The expansions can therefore be used to obtain accurate approximations to the percentage points of the test statistics even for comparatively small values of n (as small as 10).

A discussion of the practical applications of the above tests will now be considered. One example involves life testing experiments where the $\theta_i$'s represent either the guaranteed minimum operational time of an electronic part, the normal minimum interval between accidents because of increased safety consciousness/mandatory work stoppage, or some circumstance that eliminates the possibility of simultaneous events occurring. The different population $\theta_i$'s (location parameters) can now be tested for equivalence. A second example consists of testing the same equipment on several aircraft. The interval between successive failures of this equipment in operating hours is gathered for each aircraft. A maintenance application considers whether the failure rate differs significantly among the different aircraft. One of the test statistics from the three hypotheses can accomplish this. A third example entails comparing the failure rate of several similar electronic components. A test is conducted and the failure times of each of the different samples of electronic components are recorded. To test if the components have the same average failure rate, the null hypothesis $H_o$ is appropriate. Null Hypothesis $H_o$ tests to determine if the samples

are identical with the same scale parameter (expected time between events/failure time) and the same location parameter (minimum interval between events/minimum failure time/guaranteed minimum operation time).

If different information is required on the different populations, Hypotheses $H_1$ or $H_2$ might be appropriate. Null Hypothesis $H_1$ tests to determine if the scale parameters of each of the samples are identical, but does not care what values the location parameter assumes. Null Hypothesis $H_2$ can be used to determine if one storage area/condition or installation technique had an effect on the electronic components in their minimum failure time (location parameter) when it is known that the failure rate (scale parameter) will remain constant once installed.

The criteria derived for each of the hypotheses is based on the likelihood ratio test of Neyman and Pearson. For each of the hypotheses $H_o$, $H_1$, and $H_2$, their associated criteria are $\lambda_o$, $\lambda_1$, and $\lambda_2$, respectively, and these are discussed in Chapter III.

## Background

The examination of several populations with underlying exponential distributions to determine if they were equivalent was initially studied by P.V. Sukhatme in 1936 in his paper "On the Analysis of K Samples From Exponential Populations With Especial Reference to the Problem of Random Intervals" (Ref 18). This thesis is an extension of Sukhatme's 1936 paper. In 1941, E. Paulson discussed the power functions and the question of bias of some of the likelihood - ratio tests dealing with the exponential distribution (Ref 16). Maguire, Pearson, and Wynn argued that if one is trying to have the earliest detection of possible changes in the expectation of accidents it is better to use the interval

between accidents with the exponential distribution instead of the frequency of accidents in a given time period with the poisson distribution (Ref 13). Epstein and Tsao in 1953, took special cases of Sukhatme's LRT for two exponential populations and reduced these tests to equivalent tests which are expressed in terms of CHI Square and F distributions (Ref 5). Epstein and Sobel in their papers "Life Testing" and "Some Theorems Relevant to Life Testing From an Exponential Distribution" propose that the exponential distribution is appropriate in dealing with electronic components and that when life testing a sample of components, every component does not need to be destroyed to get a good estimate of the scale parameter (Ref 7 and 6). Zelen expands the life testing methods to include the analyzing of data when the data is taken over a range of different environmental conditions (Ref 21). Epstein justifies the validity of using the exponential distribution in additional papers and gives tests to determine if the underlying distribution is exponential and how to determine the validity of the parameters (Ref 3 and 4). In 1963, Hogg and Tanis demonstrated how an iterated procedure could be used to test the hypotheses considered in this thesis. This procedure allows the analyst to determine which observations caused the hypothesis to fail (Ref 9). Weinmann, Dugger, Franck, and Hewett compare the Kumar and Patel test with the Epstein and Tsao test and give suggestions on which test is more appropriate when testing for two exponential distributions for equivalent location parameters (Ref 19). Further results on distribution theory associated with two parameter exponential populations can be found in Hseih (Ref 10) and Mann, Shaffer, and Singpurwalla (Ref 14).

## Objective

The aim of this thesis is to:

1. Derive asymptotic expansions of the distributions of the three test statistics $\lambda_0$, $\lambda_1$, and $\lambda_2$ which are valid for moderately small values of $N(N>10)$.

2. Prepare tables of percentage points for each test statistic for $\alpha = .10, .05, .025, .01, .005$ and for $N = 10$ to $100$.

3. Illustrate the proper technique in applying the results of this thesis to actual data.

Chapter II provides preliminary information on one and two parameter exponential distributions, the Gamma distribution, random variables drawn from exponential distributions, and the Neyman and Pearson likelihood ratio criterion. Chapter III derives the criteria for the three hypotheses while Chapter IV derives the $h^{th}$ moment of the criteria. Using the moments of these criteria, Chapter V finds the asymptotic expansion of the distributions of the test statistics. Chapter VI uses actual data to demonstrate the practical application of the theoretical results obtained in this thesis.

## II. Statistical Preliminaries

Chapter II provides preliminary information on one and two parameter exponential distributions, the relationship between each other and with the gamma distribution. In addition, the distributions of random variables drawn from exponential distributions are derived.

### Two Parameter Exponential Distribution ($\theta,\sigma$)

The p.d.f. of a two parameter exponential distribution with parameters $\theta$ and $\sigma$, where $\theta$ is the location parameter and $\sigma$ is the scale parameter is

$$p(x,\theta,\sigma) = \begin{cases} \frac{1}{\sigma} \exp\left[\frac{-(x-\theta)}{\sigma}\right] & , x>\theta,\ \sigma>0,\ \theta>0 \\ 0 & , \text{otherwise} \end{cases} \tag{2.1}$$

By definition a p.d.f. equals one when evaluated over its parameter space and so.

$$\int_0^\infty \frac{1}{\sigma} \exp\left[\frac{-(x-\theta)}{\sigma}\right] dx = 1$$

Multiplying through by $\sigma$ yields

$$\int_0^\infty \exp\left[\frac{-(x-\theta)}{\sigma}\right] dx = \sigma \tag{2.2}$$

### One Parameter Exponential Distribution ($\sigma$)

The p.d.f. of a one parameter exponential distribution with parameter $\sigma$, where $\sigma$ is the scale parameter is

$$p(x,\sigma) = \begin{cases} \frac{1}{\sigma} \exp\left[\frac{-x}{\sigma}\right] & , x > 0,\ \sigma > 0 \\ 0 & , \text{otherwise} \end{cases} \tag{2.3}$$

By definition of p.d.f.,

$$\int_0^\infty \frac{1}{\sigma} \exp\left[\frac{-x}{\sigma}\right] dx = 1$$

Multiplying through by $\sigma$

$$\int_0^\infty \exp\left[\frac{-x}{\sigma}\right] dx = \sigma \tag{2.4}$$

<u>Gamma Distribution ($\alpha,\beta$)</u>

The p.d.f. of the gamma distribution is

$$\text{Gamma}(\alpha,\beta) = \begin{cases} \dfrac{X^{\alpha-1} \exp\left[\frac{-x}{\beta}\right]}{\beta^\alpha\, \Gamma(\alpha)} & , x>0, \beta>0, \alpha>0 \\ 0 & , \text{otherwise} \end{cases} \tag{2.5}$$

By definition of a p.d.f.

$$\int_0^\infty \frac{X^{\alpha-1} \exp\left[\frac{-x}{\beta}\right]}{\beta^\alpha\, \Gamma(\alpha)} dx = 1$$

The above equation is equivalent to

$$\int_0^\infty X^{\alpha-1} \exp\left[\frac{-X}{\beta}\right] dx = \beta^\alpha\, \Gamma(\alpha) \tag{2.6}$$

## Relationship Between Two Parameter and One Parameter Exponential Distributions

The following theorem shows the relationship between the two parameter and one parameter exponential distribution:

Theorem 2.1: If X has an exponential p.d.f. $p(x,\theta,\sigma)$ defined in equation (2.1), then $X-\theta$ has a one parameter exponential p.d.f. $p(x,\sigma)$ given in equation (2.3).

Proof: Let $y=x-\theta$

From hypothesis $X \sim p(x,\theta,\sigma)$

where $p(x,\theta,\sigma)$ is exponential distribution described by equation 2.1.

The following lemma is required:

Lemma 2.1: Given the p.d.f. of X, then the p.d.f. of Y, where $g(X)=Y$, is $p(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$ (Ref 2:245)

It follows from first assumption that $x=y+\theta$

and $\left|\frac{d}{dy}g^{-1}(y)\right| = 1$

Therefore the p.d.f. of Y is

$$g(y) = p(x,\theta,\sigma)\cdot\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{1}{\sigma}\exp\left[\frac{-(x-\theta)}{\sigma}\right]\cdot|1|$$

$$= \frac{1}{\sigma}\exp\left[\frac{-(y+\theta-\theta)}{\sigma}\right] = \frac{1}{\sigma}\exp\left[\frac{-y}{\sigma}\right]$$

$$g(y) = \begin{cases} \frac{1}{\sigma}\exp\left[\frac{-y}{\sigma}\right] & , y>o\ ,\ \sigma>o \\ 0 & , \text{otherwise} \end{cases}$$

Relationship Between Gamma and One Parameter Exponential Distribution

A gamma distribution with $\alpha=1$ and $\beta=\sigma$ is equivalent to a one parameter distribution with parameter $\sigma$ or

$$p(x,\sigma) = \text{gamma}(1,\sigma)$$

Properties of Gamma Distribution

Moment Generating Function: The moment generating function for the gamma distribution is

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}\, p(x)dx$$

$$= \int_0^{\infty} e^{tx}\, \frac{x^{-\alpha} \exp\left[\frac{-x}{\beta}\right]}{\beta^{\alpha}\Gamma(\alpha)}\, dx = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} \cdot \exp\left[-x\left[\frac{1}{\beta}-t\right]\right] dx$$

Using equation 2.6 this is equivalent to

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\left[\frac{1}{\beta}-t\right]^{\alpha}} = \frac{1}{(1-\beta t)^{\alpha}} = (1-\beta t)^{-\alpha}$$

Therefore the moment generating function for gamma distribution is

$$M_x(t) = (1-\beta t)^{-\alpha} \tag{2.8}$$

Using equations 2.7 and 2.8 the moment generating function of X where X has a one parameter exponential p.d.f. is

$$M_x(t) = (1-\sigma t)^{-1} \tag{2.9}$$

Lemma 2.2: If $M_x(t)$ is the M.G.F. of X, then

$$M_{ax+b}(t) = e^{tb}M_x(at)$$

Proof: $M_{ax+b}(t) = E(e^{t(ax+b)}) = E(e^{tax+tb})$

$= E(e^{tax} \cdot e^{tb}) = e^{tb} \cdot E(e^{tax})$

$= e^{tb} \cdot M_x(at)$

The moment generating function of X where X has a two parameter exponential p.d.f. is

$$M_x(t) = e^{t\theta}(1-\sigma t)^{-1} \quad (2.10)$$

Proof: Let $Y=X-\theta$, Y therefore has a one parameter exponential distribution

from equation 2.9 $M_y(t) = (1-\sigma t)^{-1}$

since $X=Y+\theta$, combine Lemma 2.2 and equation 2.9 to get

$$M_{y+\theta}(t) = e^{t\theta}(1-\sigma t)^{-1}$$

<u>Theorem 2.2</u>: Suppose $X_1, X_2, \ldots, X_n$ are independent and $X_i$ has M.G.F. $M_i$, $i=1,2,\ldots,n$. Then the M.G.F. of $a_1X_1 + a_2X_2+\ldots+a_nX_n$

$$M_{a_1X_1+a_2X_2+\ldots+a_nX_n}(t) = M_1(a_1t) \; M_2(a_2t) \; \ldots \; M_n(a_nt)$$

Proof: (Ref 2:258)

<u>Theorem 2.3</u>: If $X_1,X_2,\ldots,X_n$ are independent, each with a Gamma $(a_i,\beta)$ p.d.f., then the p.d.f. of $X_1+X_2+\ldots+X_n$ is Gamma $(a_1+a_2+\ldots+a_n, \beta)$.

Proof: From Theorem 2.2

$$M_{X_1+X_2+\ldots+X_n}(t) = M_{x_1}(t) \; M_{x_2}(t) \; \ldots \; M_{x_n}(t)$$

$$=(1-\beta t)^{-a_1}(1-\beta t)^{-a_2} \ldots(1-\beta t)^{-a_n}$$

$$=(1-\beta t)^{-(a_1+a_2+\ldots+a_n)}$$

Therefore $X_1+X_2+\ldots+X_n \sim \text{Gamma}(a_1+a_2+\ldots+a_n, \beta)$

Theorem 2.4: If $X \sim$ Gamma $(a,\beta)$, then $cX \sim$ Gamma $(a,c\beta)$

Proof: If the random variable $X \sim$ Gamma $(a,\beta)$ then it's M.G.F.

$M_x(t) = (1-\beta t)^{-a}$

Applying Lemma 2.2 to equation

$M_{cX}(t)$ yields $M_x(ct)$

which is equivalent to $(1-\beta ct)^{-a}$

and is the M.G.F. of a Gamma $(a,c\beta)$

Therefore $cX \sim$ Gamma $(a,c\beta)$

Corollary 2.4.1: If X is a one parameter exponential distribution, then $cX \sim$ Gamma $(1,c\sigma)$.

Distribution Function of One and Two Parameter Exponential Distributions

A One parameter exponential distribution has the following p.d.f. and c.d.f.

$$f(x) = \begin{cases} \frac{1}{\sigma} \exp\left[\frac{-x}{\sigma}\right] & , \ x>o,\ g>o \\ 0 & , \text{ otherwise} \end{cases}$$

$$F(x) = \int_{-\infty}^{X} f(x)dx = \int_{0}^{x} \exp\left[\frac{-x}{\sigma}\right] \cdot \frac{1}{\sigma}\, dx \quad = -\int_{0}^{x} \exp\left[\frac{-x}{\sigma}\right] \cdot \left[-\frac{1}{\sigma}\right] dx$$

$$= -\exp\left[\frac{-x}{\sigma}\right] \Bigg|_{o}^{x} \quad = 1 - \exp\left[\frac{-x}{\sigma}\right]$$

A Two parameter exponential distribution has the following p.d.f. and c.d.f.

$$f(x) = \begin{cases} \frac{1}{\sigma} \quad \exp\left[\frac{-(x-\theta)}{\sigma}\right] & , x>\theta,\ \theta>o,\ \sigma>o \\ 0 & , \text{otherwise} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{\theta}^{x} \frac{1}{\sigma} \exp\left[\frac{-(x-\theta)}{\sigma}\right] dx$$

$$= \int_{\theta}^{x} \frac{1}{\sigma} \exp\left[\frac{-x}{\sigma}\right] \cdot \exp\left[\frac{\theta}{\sigma}\right] dx \ = \exp\left[\frac{\theta}{\sigma}\right] \int_{\theta}^{x} \frac{1}{\sigma} \exp\left[\frac{-x}{\sigma}\right] dx$$

$$= -\exp\left[\frac{\theta}{\sigma}\right] \cdot \left[\exp\left[\frac{-x}{\sigma}\right]\right]\Bigg|_{\theta}^{x} = \ -\exp\left[\frac{\theta}{\sigma}\right] \cdot \left[\exp\left[\frac{-x}{\sigma}\right] - \exp\left[\frac{-\theta}{\sigma}\right]\right]$$

$$= 1 - \exp\left[\frac{-(x-\theta)}{\sigma}\right]$$

<u>THEOREM</u> <u>2.5</u>: Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the order statistics from a population with a p.d.f. $p(X,\theta,\sigma)$ as defined in equation (2.1), then the p.d.f. of $X_{(1)}$ is $p(X,\theta,\sigma/n)$.

Proof: $P(X_{(1)} < x) = 1-(1-F_x(x))^n = 1-(1-P(X<x))^n$

$$= 1-\left[1-1+\exp\left[\frac{-(x-\theta)}{\sigma}\right]\right]^n = 1 - \exp\left[\frac{-n(x-\theta)}{\sigma}\right]$$

Take the derivative and the p.d.f. of $X_{(1)}$ is

$$\frac{n}{\sigma} \exp\left[\frac{-n(x-\theta)}{\sigma}\right] = \frac{1}{\sigma/n} \quad \exp\left[\frac{-(x-\theta)}{\sigma/n}\right]$$

It therefore follows that the order statistic $X_{(1)}$ has a two parameter exponential distribution with parameters of $\theta$ and $\sigma/n$. (2.11)

<u>THEOREM 2.6</u>: Let $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ be the order statistics from a population with a p.d.f. $p(x,\theta)$ defined in equation (2.3), then the p.d.f. of $X_{(1)}$ is $p(X,\sigma/n)$.

Proof: $P(X_{(1)} < x) = 1-(1-F_x(x))^n = 1-(1-1+\exp\left[\frac{-x}{\sigma}\right])^n$

$$= 1 - \exp\left[\frac{-nx}{\sigma}\right] = 1 - \exp\left[\frac{-x}{\sigma/n}\right]$$

Once again take the derivative and the p.d.f. of $X_{(1)}$ is

$$\frac{1}{\sigma/n} \exp\left[\frac{-x}{\sigma/n}\right]$$

It has been shown that $X_{(1)}$ has a one parameter exponential distribution with parameter $\sigma/n$. (2.12)

<u>THEOREM 2.7</u>: Let $X_1, X_2, \dots, X_n$ denote a random sample from a distribution having a p.d.f. $f(x:\theta)$, $\theta\epsilon\Omega$, where $\Omega$ is an interval set. Let $Y_1 = u_1(X_1,X_2,\dots,X_n)$ be a sufficient statistic for $\theta$, and let the family $\{g_1(y_1;\theta); \theta\epsilon\Omega\}$ of probability density functions of $Y_1$ be complete. Let $Z=u(X_1,X_2,\dots,X_n)$ be any other statistic (not a function of $Y_1$ alone). If the distribution of Z does not depend upon $\theta$, then Z is stochastically independent of the sufficient statistic $Y_1$.

Proof: (Ref 8:390)

<u>Corollary 2.7.1</u>: If the M.L.E. of $\theta$ is $X_{(1)}$ and the M.L.E. of $\sigma$ is $\bar{X}-X_{(1)}$, then $X_{(1)}$ and $\bar{X}-X_{(1)}$ are independent.

Theorem 2.8: The M.G.F. of $\overline{X}-X_{(1)}$ is $(1-\sigma/n\ t)^{-(n-1)}$ and the p.d.f. of $\overline{X}-X_{(1)}$ is Gamma $(n-1, \sigma/n)$, where $X_i \sim p(x,\theta,\sigma)$.

Proof: Order a sample $x_1, x_2, \ldots, x_n$ so that $x_{(1)} < x_{(2)} < \ldots < X_{(n)}$, therefore

$$\sum_{i=1}^{n} (X_{(i)}) = \sum_{i=1}^{n} X_i$$

and

$$\sum_{i=1}^{n} (X_{(i)}-\theta) = \sum_{i=1}^{n} (X_{(i)}-X_{(1)}) + n(X_{(1)}-\theta)$$

$$= \sum_{i=1}^{n} (X_i-\theta) \qquad \text{(I)}$$

By Theorem 2.1 and equation 2.7 $X_i-\theta \sim \text{Gamma}(1,\sigma)$. (2.13)

Using Theorem 2.3 and that each of the $X_i$ are independent; it follows that

$$\sum_{i=1}^{n} (X_i-\theta) \sim \text{Gamma}(n,\sigma) \qquad (2.14)$$

Therefore, by Theorem 2.2 the M.G.F. of $\sum_{i=1}^{n} (X_i-\theta) = (1-\sigma t)^{-n}$

If the order statistic $X_{(1)}$ is from a two parameter exponential population with parameters $(\theta,\sigma)$, then by Theorem 2.1 $X_{(i)}-\theta$ has a one parameter exponential distribution and by equation 2.12 its parameter is $(\sigma/n)$ which yields a Gamma $(1,\sigma/n)$ distribution. Therefore,

$$X_{(1)}-\theta \sim \text{Gamma}(1,\sigma/n) \qquad (2.15)$$

Using Theorem 2.4 $n(X_{(1)}-\theta)$ is distributed as

$$\text{Gamma}\ (1,\ (\sigma/n)\ n) = \text{Gamma}\ (1,\sigma) \qquad (2.16)$$

Therefore, the M.G.F. of $n(X_{(1)}-\theta)$ is $(1-\sigma t)^{-1}$ (2.17)

$\sum_{i=1}^{n} (X_{(i)}-X_{(1)})$ and $n(x_{(1)}-\theta)$ are now shown to be independent.

$$\sum_{i=1}^{n} (X_{(i)}-X_{(1)}) = \sum_{i=1}^{n} X_i - nX_{(1)} = n\,[\,\overline{X}-X_{(1)}]$$

As a result of Corollary 2.7.1 $\overline{X}-X_{(1)}$ and $X_{(1)}$ are independent, therefore the distributions of $\sum_{i=1}^{n} (X_i-X_{(1)})$ and $n(X_{(1)}-\theta)$ are independent. (II)

Using statements (I) and (II),

$$M_{\sum_{i=1}^{n}(X_i-\theta)}(t) = M_{\sum_{i=1}^{n}(X_{(i)}-X_{(1)}) + n(X_{(1)}-\theta)}(t)$$

and by Theorem 2.3 this equals

$$M_{\sum_{i=1}^{n} (X_{(i)}-X_{(1)}}(t) \cdot M_{n(X_{(1)}-\theta)}(t)$$

Substituting in the known M.G.F., one has

$$(1-\sigma t)^{-n} = M_{\sum_{i=1}^{n} (X_{(i)}-X_{(1)})}(t) \cdot (1-\sigma t)^{-1}$$

and so

$$M_{\sum_{i=1}^{n} (X_{(i)}-X_{(1)})}(t) = \frac{(1-\sigma t)^{-n}}{(1-\sigma t)^{-1}} = (1-\sigma t)^{-(n-1)}$$

Therefore, $\sum_{i=1}^{n} (X_{(i)}-X_{(1)}) = n(\bar{X}-X_{(1)})$ has a M.G.F. of $(1-\sigma t)^{-(n-1)}$ and is distributed as Gamma(n-1,σ). Applying Theorem 2.4,

$\bar{X}-X_{(1)} \sim$ Gamma(n-1, σ/n) with a resulting M.G.F. of

$$(1-(\sigma/n)\cdot t)^{-(n-1)}. \qquad (2.18)$$

<u>Maximum Likelihood Estimators of θ and σ for a Two Parameter Exponential Distribution</u>

The M.L.E. of θ and σ is derived using the following two definitions.

<u>Definition 1</u>: The likelihood function L, with sample values $x_1, x_2, \ldots, x_n$ is $L(X_1, X_2, \ldots, X_n, \theta, \sigma) = f(X_1, \theta, \sigma) f(X_2, \theta, \sigma) \ldots f(X_n, \theta, \sigma)$ where $f(X_i, \theta, \sigma)$ is the p.d.f. for two parameter exponential distribution (Ref:15:269).

<u>Definition 2</u>: The M.L.E. of (θ,σ) are based on a random sample $X_1, X_2, \ldots, X_n$. The (θ, σ) that maximizes $L(X_1, X_2, \ldots, X_n, \theta, \sigma)$ are the M.L.E. (θ,σ). In other words, choose θ & σ such that $L(X_1, X_2, \ldots, X_n, \theta, \sigma)$ or $\log L(X_1, X_2, \ldots, X_n, \theta, \sigma)$ is maximized (Ref 2:270).

Given a sample from a two parameter exponential distribution where each observation $x_i$ is independent, then the p.d.f. for each $x_i$ is

$$f(x_i, \theta, \sigma) = \left[\frac{1}{\sigma}\right] \exp \left[\frac{-(x_i-\theta)}{\sigma}\right] \quad x_i > \theta,\ \theta > 0,\ \sigma > 0$$

To find the M.L.E. of θ and σ, the likelihood function is

$$L(X_1, X_2, \ldots, X_n, \theta, \sigma) = \begin{cases} \dfrac{1}{\sigma^n} \exp\left[ -\sum_{i=1}^{n} \dfrac{(x_i-\theta)}{\sigma} \right], & x_{(i)} \geq \theta,\ \sigma > o,\ \theta_i \geq o \\ 0 & , X_{(i)} < \theta \end{cases}$$

Initially fix $\sigma > 0$, L is maximized by choosing $\theta$ as large as possible while assuring $(x_i - \theta) > 0$. If $\hat{\theta} = X_{(1)}$ is chosen, $\hat{\theta}$ will minimize the negative power of the function and force L to its maxima for $X_i$. Therefore the M.L.E. of $\theta$ is

$$\hat{\theta} = X_{(1)} \tag{2.19}$$

To find the M.L.E. of $\sigma$, take the logarithm of L.

$$\log L(X_1, X_2, \ldots, X_n, \theta, \sigma) = -n \log \sigma - \sum_{i=1}^{n} \frac{(x_i - \theta)}{\sigma}$$

take its derivative and set it equal to zero.

$$\frac{d \log L}{d\sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{(x_i - \theta)}{\sigma^2} = 0$$

Since $\sigma > 0$, multiply through by $\sigma$ and nothing is lost in the equation.

$$n = \sum_{i=1}^{n} \frac{(x_i - \theta)}{\sigma}$$

$$\sigma = \frac{\sum_{i=1}^{n} (x_i - \theta)}{n} = \frac{\sum_{i=1}^{n} x_i - \theta}{n} = \bar{X} - \theta$$

The M.L.E. of $\theta = X_{(1)}$, substituting in $X_{(1)}$ the M.L.E. of $\sigma$ is

$$\hat{\sigma} = \bar{X} - X_{(1)} \tag{2.20}$$

Defining Criteria $\lambda$

Definition 3: Let $X_1$, $X_2$, ..., $X_n$ denote n mutually stochastically independent random variables having the p.d.f. $f_i(x_i, \theta_1, \theta_2, \ldots, \theta_m)$, $i=1,2,\ldots,n$. The set that consists of all parameter points $(\theta_1, \theta_2, \ldots, \theta_m)$ is denoted by $\Lambda$. Let $\omega$ be a subset of the parameter spac $\Lambda$. To test the hypothesis $H_0:(\theta_1, \theta_2, \ldots, \theta_m) \; \varepsilon\omega$ against alternative hypotheses. Define the likelihood functions

$$L(\omega) = \prod_{i=1}^{n} f_i(X_1;\ \theta_1,\ \theta_2,\ \ldots,\ \theta_m) \quad , \quad (\theta_1,\ \theta_2,\ \ldots,\ \theta_m) \ \varepsilon\omega,$$

and

$$L(\Lambda) = \prod_{i=1}^{n} f_i(X_i;\ \theta_1,\ \theta_2,\ \ldots,\ \theta_m) \quad , \quad (\theta_1,\ \theta_2,\ \ldots,\ \theta_m) \ \varepsilon\Lambda$$

Let $L(\omega_o)$ and $L(\Lambda_o)$ be the maxima of these two likelihood functions. The ratio of $L(\omega_o)$ to $L(\Lambda_o)$ is the likelihood ratio and is denoted by

$$\Lambda = \frac{\max_{\theta\varepsilon\omega} L(\omega_o)}{\max_{\theta\varepsilon\Lambda} L(\Lambda_o)} \tag{2.2}$$

Let $\lambda_o$ be a positive proper fraction. The likelihood ratio test principle states that the hypothesis $H_o:(\theta_1, \theta_2, \ldots, \theta_m) \ \varepsilon\omega$ is rejected if and only if $\lambda < \lambda_o$ (Ref 8:260,26

## III. Derivation of the Criteria

In this chapter, the three likelihood ratio criteria $\lambda_o$, $\lambda_1$, $\lambda_2$ will be obtained for the three hypotheses stated in Chapter II. To get a better understanding of the overall problem, Figure 3.1 will illustrate the different notations and populations.

| | |
|---|---|
| $1^{st}$ Pop | $i^{th}$ Pop |
| $X_{11}$<br>$X_{12}$<br>$\vdots$<br>$X_{1n}$ | $X_{i1}$<br>$X_{i2}$<br>$\vdots$<br>$X_{in}$ |
| $Exp(\theta_1, \sigma_1)$ | $Exp(\theta_i, \sigma_i)$ |
| $X_{1(1)}$ is the smallest order statistic of $1^{st}$ Pop | $X_{i(1)}$ is the smallest order statistic of $i^{th}$ Pop |
| From equation (2.11)<br>M.L.E. of $\theta_1 = X_{1(1)}$ | M.L.E. of $\theta_i = X_{i(1)}$ |
| From equation (2.16)<br>$(X_{1(1)} - \theta_1)$ Gamma$(1, \sigma_1/n)$ | $(X_{i(1)} - \theta_i)$ Gamma$(1, \sigma_i/n)$ |
| From equation (2.12)<br>M.L.E. of $\sigma_1 = (\bar{X}_1 - X_{1(1)})$ | M.L.E. of $\sigma_i = (\bar{X}_i - X_{i(1)})$ |
| From Theorem 2.6<br>$(\bar{X}_1 - X_{1(1)})$ Gamma$(n-1, \sigma_1/n)$ | $(\bar{X}_i - X_{i(1)})$ Gamma$(n-1, \sigma_i/n)$ |

Figure 3.1. Sample Populations and Their Statistics

In Figure 3.1 "i" ranges from one to p, where p is the number of different samples. Each sample has same number of observations n, where n > 10.

Notations:

Let $X_{(1)}$ be the minimum of

$$X_{1(1)}, X_{2(1)}, \ldots, X_{i(1)}, \ldots, X_{p(1)} \qquad \text{(Def 3.1)}$$

$$\overline{X} = \frac{1}{p} \sum_{i=1}^{p} \overline{X}_i , \qquad \text{(Def 3.2)}$$

and

$$\overline{X}_i = \sum_{j=1}^{n} \frac{X_{ij}}{n} \qquad \text{(Def 3.3)}$$

Likelihood Functions.

Since all of the sample observations within each population are independent, the likelihood function for the first sample is

$$L_1 = f(X_{11}, \theta_1, \sigma_1) \cdot f(X_{12}, \theta_1, \sigma_1) \cdot \ldots \cdot f(X_{1n}, \theta_1, \sigma_1)$$

$$= \frac{1}{(\sigma_1)^n} \exp \left\{ \frac{-\sum_{j=1}^{n} (X_{1j} - \theta_1)}{\sigma_1} \right\}$$

In general, the likelihood function $L_i$ is

$$L_i = f(X_{i1}, \theta_i, \sigma_i) \cdot f(X_{i2}, \theta_i, \sigma_i) \cdot \ldots \cdot f(X_{in}, \theta_i, \sigma_i)$$

$$= \frac{1}{(\sigma_i)^n} \exp \left\{ \frac{-\sum_{j=1}^{n} (X_{ij} - \theta_j)}{\sigma_j} \right\} \qquad (3.1)$$

Each of the p populations are independent and the likelihood function $L(\Lambda)$ for all np sample values is given by

$$L(\Lambda)=f(X_{11},X_{12},\ldots,X_{np},\theta_1,\ldots,\theta_p,\sigma_1,\ldots,\sigma_p)$$

$$=L_1 \cdot L_2 \cdot \ldots \cdot L_p \tag{3.2}$$

$$=\prod_{i=1}^{p} \frac{1}{(\sigma_i)^n} \exp\left\{\frac{-\sum_{j=1}^{n}(X_{ij}-\theta_i)}{\sigma_i}\right\} \tag{3.3}$$

Derivation of $\lambda_o$

The first of the three criteria $\lambda_o$ is determined using the $H_o$ which states that

$\theta=\theta_1=\theta_2=\ldots=\theta_p$ and

$\sigma=\sigma_1=\sigma_2=\ldots=\sigma_p$

The hypothesis tests to determine if all of the location and the scale parameters of each of the "p" populations are equal. Let $L(\omega_o)$ be the likelihood function under $H_o$ as given in (Def 2.3). Under $H_o$, $\theta_i=\theta$ and $\sigma_i=\sigma$ the likelihood function of equation 3.6 becomes

$$L(\omega_o) = \prod_{i=1}^{p} \frac{1}{(\sigma)^n} \exp\left\{\frac{\sum_{j=1}^{n}(X_{ij}-\theta)}{\sigma}\right\}$$

Using equations 2.19 and 2.20 and the definitions 3.1 and 3.2 $L(\omega_o)$ is a maximum when $\theta=X_{(1)}$ and $\sigma=\overline{X}-X_{(1)}$. Therefore the max $\theta\varepsilon\omega_o$ $L(\omega_o)$ for equation 2.21 is

$$\prod_{i=1}^{p} \frac{1}{(\overline{X}-X_{(1)})^n} \exp\left\{\frac{-\sum_{j=1}^{n}(X_{ij}-X_{(1)})}{\overline{X}-X_{(1)}}\right\}$$

$$= \prod_{i=1}^{p} \frac{1}{(\bar{X}-X_{(1)})^n} \exp\left[\frac{-n(\bar{X}_i-X_{(1)})}{\bar{X}-X_{(1)}}\right]$$

$$= \frac{1}{(\bar{X}-X_{(1)})^{np}} \exp\left[\frac{-n}{\bar{X}-X_{(1)}} \cdot \sum_{i=1}^{p} (X_i-X_{(1)})\right]$$

using Def 3.2 yields

$$= \frac{1}{(\bar{X}-X_{(1)})^{np}} \exp\left[\frac{-n}{\bar{X}-X_{(1)}} \; p(\bar{X}-X_{(1)})\right]$$

$$= \frac{1}{(\bar{X}-X_{(1)})^{np}} \exp(-np)$$

$$\max_{\theta\varepsilon\omega} L(\omega_o) = \frac{\exp(-np)}{(\bar{X}-X_{(1)})^{np}} \qquad (3.4)$$

When finding the likelihood function for the denominator of the likelihood ratio test as defined in definition three of chapter 2, the maximum that $L(\Lambda)$ can achieve occurs when $\theta_i=X_{i(1)}$ and $\sigma_i=\bar{X}_i-X_{i(1)}$ for each population $(i=1,\ldots,p)$. Therefore, the max $\theta\varepsilon\Lambda\, L(\Lambda)$ for equation 2.1 is

$$\prod_{i=1}^{p} \frac{1}{(\bar{X}_i-X_{i(1)})^n} \exp\left\{\frac{-\sum_{j=1}^{n} (X_{ij}-X_{i(1)})}{X_i-X_{i(1)}}\right\}$$

$$= \frac{1}{\prod_{i=1}^{p} (\bar{X}_i-X_{i(1)})^n} \exp\left\{-\sum_{i=1}^{p} \frac{\sum_{j=1}^{n} (X_{ij}-X_{(1)})}{X_i-X_{i(1)}}\right\}$$

$$= \frac{1}{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^n} \exp\left\{ -\sum_{i=1}^{p} \frac{n(\bar{X}_i - X_{i(1)})}{\bar{X}_i - X_{i(1)}} \right\}$$

$$= \frac{\exp(-np)}{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^n}$$

$$\max_{\theta\varepsilon\Lambda} L(\Lambda) = \frac{\exp(-np)}{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^n} \tag{3.5}$$

From equation 2.21, criteria $\lambda_o$ which satisfies the assumptions under $H_o$ is

$$\frac{\max_{\theta\varepsilon\omega} L(\omega_o)}{\max_{\theta\varepsilon\Lambda} L(\Lambda)} = \frac{\dfrac{\exp(-np)}{(\bar{X}-X_{(1)})^{np}}}{\dfrac{\exp(-np)}{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^n}}$$

$$\frac{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^n}{(\bar{X}-X_{(1)})^{np}} = \frac{\prod_{i=1}^{p} \ell_i^{\,n}}{(\ell_o)^{np}} \tag{3.6}$$

where $\ell_i = \bar{X}_i - X_{i(1)}$ (3.7)

and $\ell_o = \bar{X} - X_{(1)}$ (3.8)

Using equation 2.18 $\ell_i \sim \text{Gamma}(n-1,\ \sigma_i/n)$

Derivation of $\lambda_1$

The second of the three criteria $\lambda_1$ is determined using the $H_1$ (hypothesis) which states

$\sigma=\sigma_1=\sigma_2=\ldots=\sigma_p$ and

$\theta_i$'s unspecified $(i=1,2,\ldots,p)$

This hypothesis tests to determine if each of the scale parameters are equal for all "p" populations. The likelihood function, for all $\sigma_i=\sigma$, becomes

$$L(\omega_1)=\prod_{i=1}^{p}\frac{1}{(\sigma)^n}\exp\left\{\frac{-\sum_{j=1}^{n}(X_{ij}-\theta_i)}{\sigma}\right\}$$

$$=\frac{1}{(\sigma)^{np}}\exp\left\{\frac{-\sum_{i=1}^{p}\sum_{j=1}^{n}(X_{ij}-\theta_i)}{\sigma}\right\}$$

From chapter two and equation 2.19 the M.L.E. of $\theta_i$ is $X_{i(1)}$. To determine the point at which $L(\omega_1)$ is a maximum, take the derivative of $\log L(\omega_1)$ and set it equal to zero.

$$\log L(\omega_1) = -np\log\sigma - \frac{1}{\sigma}\sum_{i=1}^{p}\sum_{j=1}^{n}(X_{ij}-\theta_i)$$

$$\frac{d\log L(\omega_1)}{d\sigma} = \frac{-np}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{p}\sum_{j=1}^{n}(X_{ij}-\theta_i)$$

Set this equation equal to 0 and solve for $\sigma$ to find its maxima.

$$np = \frac{1}{\sigma} \sum_{i=1}^{p} \sum_{j=1}^{n} (X_{ij}-\theta_i)$$

$$\sigma = \frac{1}{np} \sum_{i=1}^{p} \sum_{j=1}^{n} (X_{ij}-\theta_i)$$

Substituting the M.L.E. of $\theta_i$ into the equation, the M.L.E. of $\sigma$ is given by

$$= \frac{1}{np} \sum_{i=1}^{p} \sum_{j=1}^{n} (X_{ij}-X_{i(1)}) \tag{3.9}$$

$$= \frac{1}{np}\left[\sum_{i=1}^{p} \sum_{j=1}^{n} X_{ij} - n \sum_{i=1}^{p} X_{i(1)}\right] = \frac{1}{p} \sum_{i=1}^{p} \bar{X}_i - \frac{1}{p} \sum_{i=1}^{p} X_{i(1)}$$

$$= \frac{1}{p} \sum_{i=1}^{p} (\bar{X}_i - X_{i(1)}) = \frac{1}{p} \sum_{i=1}^{p} \ell_i$$

where $\ell_i$ is defined in equation 3.7.

Thus the M.L.E. of $\sigma = \frac{1}{p} \sum_{i=1}^{p} \ell_i$ (3.10)

Using equations 2.19, 3.9, and 3.10 the denominator of criteria $\lambda_1$ is

$$\max_{\theta\epsilon\omega_1} L(\omega_1) = \frac{1}{\left[\frac{1}{p} \sum_{i=1}^{p} \ell_i\right]^{np}} \exp\left\{\frac{-\sum_{i=1}^{p} \sum_{j=1}^{n} (X_{ij}-X_{i(1)})}{\left[\sum_{i=1}^{p} \sum_{j=1}^{n} \left[X_{ij}-X_{i(1)}\right]\right]\cdot\frac{1}{np}}\right\}$$

$$= \frac{\exp(-np)}{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^{np}}$$

The max $\theta\epsilon\Lambda$ $L(\Lambda)$ for the second hypothesis is the same as the first hypothesis in determining the likelihood ratio test and is given in equation 3.5. Criteria $\lambda_1$ which satisfies the assumptions under $H_1$ is

$$\lambda_1 = \frac{\max\limits_{\theta\epsilon\omega_1} L(\omega_1)}{\max\limits_{\theta\epsilon\Lambda} L(\Lambda)} = \frac{\dfrac{\exp(-np)}{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^{np}}}{\dfrac{\exp(-np)}{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^n}}$$

$$= \frac{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^n}{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^{np}} = \frac{\prod_{i=1}^{p} (\ell_i)^n}{\bar{\ell}^{np}} = \lambda_1 \qquad (3.11)$$

$$\text{where } \bar{\ell} = \frac{\ell_1 + \ell_2 + \ldots + \ell_p}{p} \qquad (3.12)$$

Note that $\lambda_1$ is a function of $\ell_i$ only.

Derivation of $\lambda_2$

The third criteria $\lambda_2$ is determined using hypothesis $H_2$ which states that

$\theta=\theta_1=\theta_2=\ldots=\theta_p$ knowing that

$\sigma=\sigma_1=\sigma_2=\ldots=\sigma_p$ is true.

This hypothesis tests to determine if each of the location parameters are equal for all "p" populations, knowing that all "p" populations have the same scale parameter. The likelihood function, for all $\sigma_i$'s = $\sigma$ and $\theta_i$'s = $\theta$, for $H_2$ results in the same maximum $L(\omega_2)$ as in $H_o$, from equation 3.4

$$\max_{\theta\varepsilon\omega} L(\omega_2) = \frac{\exp(-np)}{(\bar{X}-X_{(1)})^{np}}$$

The denominator of the likelihood ratio test for the third hypothesis is also tied to all of the scale parameters being equal and is therefore different from the first two hypotheses. The likelihood function is

$$L(\Lambda) = \prod_{i=1}^{p} \frac{1}{(\sigma)^n} \exp\left\{\frac{-\sum_{j=1}^{n} (X_{ij}-\theta_i)}{\sigma}\right\} \tag{3.13}$$

From equations 3.9 and 3.10 the M.L.E. of $\sigma$ is

$$\frac{1}{np} \sum_{i=1}^{p} \sum_{j=1}^{n} (X_{ij}-X_{i(1)}) = \frac{1}{p} \sum_{i=1}^{p} \ell_i$$

Applying equation 2.19 to multiple populations the M.L.E. of $\theta_i = X_{i(1)}$.

Substituting the M.L.E. of the two parameters into equation 3.16 yields

$$\max_{\theta\varepsilon\Lambda_2} L(\Lambda_2) = \prod_{i=1}^{p} \frac{1}{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^n} \exp\left\{\frac{-\sum_{j=1}^{n}(X_{ij}-X_{i(1)})}{\sum_{i=1}^{p}\sum_{j=1}^{n}(X_{ij}-X_{i(1)})}\cdot np\right\}$$

$$= \frac{1}{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^{np}} \exp\left\{\frac{-\sum_{i=1}^{p}\sum_{j=1}^{n}(X_{ij}-X_{i(1)})}{\sum_{i=1}^{p}\sum_{j=1}^{n}(X_{ij}-X_{i(1)})}\cdot np\right\}$$

$$= \frac{\exp(-np)}{\left(\frac{1}{p}\sum_{i=1}^{p} \ell_i\right)^{np}}$$

Criteria $\lambda_2$ which satisfies the assumptions under $H_2$ is

$$\lambda_2 = \frac{\max_{\theta\varepsilon\omega_2} L(\omega_2)}{\max_{\theta\varepsilon\Lambda_2} L(\Lambda_2)} = \frac{\dfrac{\exp(-np)}{(\bar{X}-X_{(1)})^{np}}}{\dfrac{\exp(-np)}{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^{np}}}$$

$$= \frac{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^{np}}{(\bar{X}-X_{(1)})^{np}} = \frac{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^{np}}{(\ell_o)^{np}} \qquad (3.14)$$

From equation 3.8

$$\ell_o = \bar{X}-X_{(1)} = \left(\frac{1}{p}\sum_{i=1}^{p} \bar{X}_i\right) - X_{(1)}$$

using equation 3.7

$$\overline{X}_i = \ell_i + X_{i(1)} \text{ and therefore}$$

$$\ell_o = \frac{1}{p} \sum_{i=1}^{p} (\ell_i + X_{i(1)}) - X_{(1)}$$

$$= \frac{1}{p} \sum_{i=1}^{p} \ell_i + \frac{1}{p} \sum_{i=1}^{p} X_{i(1)} - X_{(1)}$$

Define $$u = \frac{1}{p} \sum_{i=1}^{p} X_{i(1)} - X_{(1)} \tag{3.15}$$

then $$\ell_o = \frac{1}{p} \sum_{i=1}^{p} \ell_i + u \tag{3.16}$$

From equations 3.14, 3.15, 3.16

$$\lambda_2 = \frac{\left[\frac{1}{p} \sum_{i=1}^{p} \ell_i\right]^{np}}{\left[\frac{1}{p} \sum_{i=1}^{p} \ell_i + u\right]^{np}} = \frac{(v)^{np}}{(v+u)^{np}} \tag{3.17}$$

where $$v = \frac{1}{p} \sum_{i=1}^{p} \ell_i \tag{3.18}$$

## IV. Derivation of the Moments

In this chapter the $h^{th}$ moments of criteria $\lambda_o$, $\lambda_1$, and $\lambda_2$ will be derived. Using these moments the sampling distribution will then be obtained in Chapter V. Prior to determining the $h^{th}$ moment of $\lambda_o$, a technique to obtain such moments will be explained.

### A Technique to Obtain $h^{th}$ Moments

Suppose we want to find

$$E\left\{\frac{\prod_{i=1}^{p} \ell_i^{nh}}{\ell_o^{nph}}\right\}$$

where $\ell_i > o$, $i=1,2,\ldots,p$

$\ell_o$ is a function of $\ell_1,\ldots,\ell_p$.

To find this let $f(\theta)$ be a function of $\theta$ defined by

$$f(\theta)=E\left[\left[\prod_{i=1}^{p} \ell_i^{nh}\right] e^{\theta\ell_o}\right]$$

$$f(\theta)=\int_{\ell_1,\ldots,\ell_p>o}\cdots\int \left[\prod_{i=1}^{p} \ell_i^{nh}\right] e^{\theta\ell_o}\, p(\ell_1,\ldots,\ell_p)d\ell_1,\ldots,d\ell_p \tag{4.1}$$

Take the first derivative on both sides of the equation and

$$\frac{df(\theta)}{d\theta}=\int_{\ell_1,\ldots,\ell_p>o}\cdots\int \left[\prod_{i=1}^{p} \ell_i^{nh}\right] e^{\theta\ell_o}\,\ell_o\, p(\ell_1,\ldots,\ell_p)d\ell_1,\ldots,d\ell_p d \tag{4.2}$$

continuing the process the $r^{th}$ derivative of equation 4.1 is

$$\frac{d}{d\theta^r}f(\theta)=\int_{\ell_1,\ldots,\ell_p>o}\cdots\int \left[\prod_{i=1}^{p} \ell_i^{nh}\right] e^{\theta\ell_o}\,\ell_o^{\,r}\, p(\ell_1,\ldots,\ell_p)d\ell_1,\ldots,d\ell_p \tag{4.3}$$

Then putting $\theta$=o and r=-nph on both sides of equation 4.3 one has

$$\left.\frac{d\, f(\theta)}{d\theta^r}\right|_{\substack{\theta=o\\ r=-nph}} = \int\cdots\int_{\ell_1,\ldots,\ell_p>o} \frac{\prod_{i=1}^{p} \ell_i^{nh}}{\ell_o^{nph}} \; p(\ell_1,\ldots,\ell_p)d\ell_1,\ldots,d\ell_p$$

$$= E\left[\frac{\prod_{i=1}^{p} \ell_i^{nh}}{\ell_o^{\,nph}}\right] \qquad (4.$$

The validity of this operation can be justified in its similarity to uses by Wilks. (5:274)

__Lemma 4.1__

Let $f(\theta) = K(1- B_1\theta)^{-\alpha}$ (4.5

If $\theta$=O, then $\frac{d}{d\theta^r} f(\theta) = K\ (\alpha)(\alpha+1)\ldots(\alpha+(r-1))\ B_1{}^r$

$$= K\ \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\ \left[B_1\right]^r \qquad (4.$$

__Theorem 4.1__: The $h^{th}$ moment of $\lambda_o$ is given by

$$E(\lambda_o^h) = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^p \cdot \frac{\Gamma(np-1)}{\Gamma(nph+np-1)} \cdot p^{nph}$$

Proof:

$$E(\lambda_o^h) = E\left\{\frac{\prod_{i=1}^{p} \ell_i^{nh}}{\ell_o^{nph}}\right\} = E\left\{\frac{\prod_{i=1}^{p} \ell_i^{nh}}{\left[\frac{1}{p}\prod_{i=1}^{p} \ell_i + u\right]^{nph}}\right\}$$

$$= \int \cdots \int \int_{\ell_1,\ldots,\ell_p > o,\, u > o} \frac{\prod_{i=1}^{p} \ell_i^{\,nh}}{\left(\frac{1}{p}\sum_{i=1}^{p} \ell_i + u\right)^{nph}} \, p(\ell_1, \ell_2, \ldots, \ell_p, u) \, d\ell_1 d\ell_2 \ldots d\ell_p du \qquad (4.7)$$

From equation 3.15, it can be shown that u is a function such that

$u=f(x_{1(1)},\ x_{2(1)},\ \ldots,\ x_{p(1)})$

Corollary 2.7.1 states that $X_{(1)}$ and $\bar{X}-X_{(1)}$ are independent. From the corollary then, it is known that for any given sample $\bar{X}_i-X_{i(1)}=\ell_i$ and $\bar{X}_{i(1)}$ are independent. Since u is a function of $X_{i(1)}$, $\ell_i$ and u are independent. Therefore equation 4.7 is equivalent to

$$\int \cdots \int \int_{\ell_1\ldots,\ell_p > o,\, u > o} \frac{\prod_{i=1}^{p} \ell_i^{\,nh}}{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i + u\right]^{nph}} \, p(\ell_1), p(\ell_2), \ldots, p(\ell_p) p(u) d\ell_1 \ldots, d\ell_p du \qquad (4.8)$$

From Chapter Two the following is true:

$\ell_i \sim \text{Gamma}(n-1,\ \sigma/n)$ equation 2.18

$u \sim \text{Gamma}(p-1,\ \sigma/np)$ (4.9)

Theorem 2.6 gives the distribution of $X_{i(1)}$ to be an exponential distribution with parameters $\theta$ and $\sigma/n$. Applying Theorem 2.6 again to a sample of size p $X_{i(1)}$ random variables yields that $u \sim \text{Gamma}(p-1,\sigma/np)$.

In order to use the technique described in the beginning of this chapter, consider a function of $\theta$ defined as follows

$$f(\theta) = E\left\{\left[\prod_{i=1}^{p} \ell_i^{\,nh}\right] \cdot e^{\theta\cdot\left[\frac{1}{p}\sum_{i=1}^{p}\ell_i + u\right]}\right\}$$

$$= \int\cdots\int\int_{\ell_1\ldots,\ell_p>o,u>o} \left[\prod_{i=1}^{p} \ell_i^{\,nh}\right]\cdot \exp\left[\frac{\theta}{p}\sum_{i=1}^{p}\ell_i + \theta\cdot u\right] p(\ell_1,\ldots,\ell_p,u)\,d\ell_1\ldots d\ell_p du$$

each of the p populations are independent and therefore

$$= \int\cdots\int\int_{\ell_1\ldots\ell_p>o,u>o} \left[\prod_{i=1}^{p} \ell_i^{\,nh}\right]\cdot \exp\left[\frac{\theta}{p}\sum_{i=1}^{p}\ell_i\right] e^{\theta u}\; p(\ell_1)\ldots p(\ell_p)p(u)\,d\ell_1\ldots d\ell_p du$$

Substituting the p.d.f. of each $p(\ell_i)$ and $p(u)$ results in the equation:

$$\int_{\ell_1\ldots\ell_p>o,u>o} \frac{\left[\prod_{i=1}^{p} \ell_i^{\,nh}\right]\exp\left[\frac{\theta}{p}\sum_{i=1}^{p}\ell_i\right]\prod_{i=1}^{p}\ell_i^{(n-1)-1}\exp\left[-\sum_{i=1}^{p}\frac{\ell_i}{\sigma/n}\right]}{(\sigma/n)^{p(n-1)}\left[\Gamma(n-1)\right]^p\;(\sigma/np)^{p-1}\;\Gamma(p-1)}$$

$$X\; e^{\theta u}\, u^{(p-1)-1}\;\exp\left[\frac{-u}{\sigma/np}\right] d\ell_1\ldots d\ell_p du$$

$$= K\cdot\int\cdots\int_{\ell_1\ldots\ell_p>o}\left[\prod_{i=1}^{p}\ell_i^{\,nh+(n-1)-1}\exp\left[\frac{\theta}{p}\ell_i - \frac{\ell i}{\sigma/n}\right]\right]d\ell_1\ldots d\ell_p$$

$$X\int_{u>o} u^{(p-1)-1}\exp\left[\theta u - \frac{u}{\sigma/np}\right]du \qquad (4.10)$$

where $K = \dfrac{1}{\left[\frac{\sigma}{n}\right]^{p(n-1)} \left[\Gamma(n-1)\right]^{p} \left[\frac{\sigma}{np}\right]^{p-1} \Gamma(p-1)}$

Using equation 2.6 definition of the Gamma distribution and the knowledge that each $\ell_i$ is a random variable from the same distribution permits replacement of the product of the $\ell_i$'s by raising the first integrand to the appropriate power.

Equation 4.10 makes the transformation to this equation

$$= K \cdot \left[\frac{\Gamma(nh+n-1)}{\frac{n}{\sigma} - \frac{\theta}{p}^{\,nh+n-1}}\right]^{p} \cdot \left[\frac{\Gamma(p-1)}{\frac{np}{\sigma} - \theta^{\,p-1}}\right] \tag{4.11}$$

Replacing K with its equation, equation 4.11 simplifies to

$$= \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^{p} \cdot \frac{\left[\frac{\sigma}{n}\right]^{nph}}{\left[1 - \frac{\sigma\theta}{np}\right]^{nph+np-1}}$$

$$= K_1 \cdot \left[1 - \frac{\sigma\theta}{np}\right]^{-\alpha} \tag{4.12}$$

where $K_1 = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^{p} \cdot \left[\frac{\sigma}{n}\right]^{nph}$

$\alpha = nph + pn - 1$

Now according to the technique

$$E(\lambda_o^{h}) = \frac{d \quad f(\theta)}{d\theta^{r}} \qquad \text{with } \theta = 0 \text{ and } r = -nph$$

where $f(\theta)$ is of the form described in Lemma 4.1.

So using equation 4.6, one has

$$\frac{d}{d\theta^r} f(\theta)\Big|_{\theta=o} = K \cdot \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\left[\frac{\sigma}{np}\right]^r$$

Substitute $\frac{\sigma}{np}$ for $B_1$ in equation 4.5 and -nph for r in equation 4.6 and

the desired results

$$E(\lambda_o^h) = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^p \cdot \frac{\Gamma(np-1)}{\Gamma(nph+np-1)} \cdot p^{nph} \qquad (4.13)$$

<u>Theorem</u> <u>4.2</u>: The $h^{th}$ moment of $\lambda_1$ is given by

$$E(\lambda_1^h) = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^p \cdot p^{nph} \cdot \frac{\Gamma(np-p)}{\Gamma(nph+np-p)}$$

Proof:

$$E(\lambda_1^h) = E\left\{\frac{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^{nh}}{\left[\frac{1}{p}\prod_{i=1}^{p} \ell_i\right]^{nph}}\right\} = E\left[\frac{\prod_{i=1}^{p} (\ell_i)^{nh}}{\bar{\ell}^{nph}}\right]$$

where $\bar{\ell}$ is defined in equation 3.12.

$$= \int_{\ell_1 \ldots \ell_p > o} \cdots \int \frac{\prod_{i=1}^{p} (\ell_i)^{nh}}{\left[\frac{1}{p}\sum_{i=1}^{p} \ell_i\right]^{nph}} p(\ell_1, \ldots, \ell_p) d\ell_1, \ldots, d\ell_p$$

Consider a second function defined for criteria $\lambda_1$ as follows

$$g(\theta) = E\left[\left[\prod_{i=1}^{p} \ell_i^{\,nh}\right] \exp\left[\frac{\theta}{p}\sum_{i=1}^{p} \ell_i\right]\right]$$

$$= \int \cdots \int_{\ell_1 \ldots \ell_p > o} \left[ \prod_{i=1}^{p} \ell_i^{nh} \right] \exp \left[ \frac{\theta}{p} \sum_{i=1}^{p} \ell_i \right] p(\ell_1) \ldots p(\ell_p) d\ell_1 \ldots d\ell_p$$

Substituting the p.d.f. of each $p(\ell_i)$ results in the equation:

$$\int \cdots \int_{\ell_1 \ldots \ell_p > o} \frac{\left[ \prod_{i=1}^{p} \ell_i^{nh} \right] \exp \left[ \frac{\theta}{p} \sum_{i=1}^{p} \ell_i \right] \prod_{i=1}^{p} \ell_i^{(n-1)-1} \exp \left[ -\sum_{i=1}^{p} \frac{\ell_i}{\sigma/n} \right]}{\left[ \frac{\sigma}{n} \right]^{p(n-1)} \left[ \Gamma(n-1) \right]^p} \quad d\ell_1 \ldots d\ell_p$$

$$= K \cdot \left[ \int_{\ell_i > o} \ell_i^{nh+n-1-1} \quad \exp \left[ -\ell_i \left[ \frac{1}{\sigma/n} - \frac{\theta}{p} \right] \right] \right]^p \quad d\ell_i \tag{4.14}$$

where $K = \dfrac{1}{\left[ \frac{\sigma}{n} \right]^{p(n-1)} \left[ \Gamma(n-1) \right]^p}$

Using equation 2.6 definition of the Gamma distribution, equation 4.14 transforms to

$$K \cdot \left[ \frac{\Gamma(nh+n-1)}{\left[ \frac{1}{\sigma/n} - \frac{\theta}{p} \right]^{nh+n-1}} \right]^p \tag{4.15}$$

Replacing K with its equation in equation 4.14 simplifies to

$$= \frac{\left[ \Gamma(nh+n-1) \right]^p \left[ \frac{\sigma}{n} \right]^{nph}}{\Gamma(n-1)^p \left[ 1 - \frac{\theta\sigma}{np} \right]^{nph+np-p}}$$

$$g(\theta) = K_2 \cdot \left[ 1 - \frac{\theta\sigma}{np} \right]^{-\alpha} \tag{4.16}$$

where $K_2 = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^p \left[\frac{\sigma}{n}\right]^{nph}$

$\alpha = nph + np - p$

Once again it can be shown that, using same arguments as for $\lambda_o$ that

$$E(\lambda_1^h) = \frac{d}{d\theta^r} g(\theta) \Big|_{\substack{\theta=o \\ r=-nph}}$$

Equation 4.16 is in the same form as equation 4.5, therefore

$$\frac{d}{d\theta^r} g(\theta) = K_2 \cdot \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \left[\frac{\sigma}{np}\right]^r$$

The $h^{th}$ moment of criteria $\lambda_1$, after substituting for $K_2$, $\alpha$, and $r$ is

$$E(\lambda_1^h) = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^p \cdot p^{nph} \cdot \frac{\Gamma(np-p)}{\Gamma(nph+np-p)}$$

<u>Theorem 4.3</u>: The $h^{th}$ moment of $\lambda_2$ is given by

$$E(\lambda_2^h) = \frac{\Gamma(nph+np-p)\,\Gamma(np-1)}{\Gamma(nph+np-1)\,\Gamma(np-p)}$$

Proof:

$$E(\lambda_2^h) = E\left[\frac{(v)^{nph}}{(v+u)^{nph}}\right]$$

$$= \int_{v>o}\int_{u>o} \frac{(v)^{nph}}{(v+u)^{nph}} \quad p(v)\ p(u)\ dvdu$$

Consider a third function defined for criteria $\lambda_2$ as follows

$$h(\theta) = E\left[v^{nph} \quad e^{\theta(u+v)}\right]$$

$$= \int_{v>o} \int_{u>o} v^{nph} \exp(\theta(u+v)) \quad p(v) \; p(u) \; dv \; du \tag{4.17}$$

The variable u is distributed Gamma (p-1, $\sigma$/np). It is known that $\ell_i \sim$ Gamma (n-1, $\sigma$/np). Using Theorem 2.4, $\ell_i/p \sim$ Gamma(n-1), $\sigma$/np). Since each $\ell_i$ comes from a different independent sample and the criteria is testing for equivalence of samples, applying Theorem 2.3 to $\ell_i/p$ will result in V being distributed Gamma(pn-p, $\sigma$/np).

Substituting in the appropriate Gamma function for the p.d.f., Equation 4.17 becomes

$$= \int_{v>o}\int_{u>o} v^{nph} \exp(\theta(v+u)) \frac{v^{pn-p-1} \exp\left[\frac{-v}{\sigma/np}\right]}{\Gamma(pn-p)\left[\frac{\sigma}{np}\right]^{pn-p}} \frac{u^{p-1-1} \exp\left[\frac{-u}{\sigma/np}\right]}{\Gamma(p-1)\left[\frac{\sigma}{np}\right]^{p-1}} \; dvdu$$

$$= K \cdot \int_{v>o} v^{nph+pn-p-1} \exp\left[-v\left[\frac{1}{\sigma/np} - \theta\right]\right] dv \int_{u>o} u^{p-1-1} \exp\left[-u\left[\frac{1}{\sigma/np} - \theta\right]\right] du \tag{4.18}$$

where $K = \dfrac{1}{\Gamma(pn-p) \; \Gamma(p-1) \; \left[\sigma/np\right]^{pn-1}}$

Using equation 2.6 definition of the Gamma function, equation 4.18 transforms to

$$h(\theta) = K\cdot\left[\frac{\Gamma(nph+np-p)}{\left[\frac{1}{\sigma/np}-\theta\right]^{nph+np-p}}\right]\left[\frac{\Gamma(p-1)}{\left[\frac{1}{\sigma/np}-\theta\right]^{p-1}}\right]$$

$$= K_3\left[1-\frac{\theta\sigma}{np}\right]^{-\alpha} \tag{4.19}$$

where $K_3 = \dfrac{\Gamma(nph+np-p)\left[\frac{\sigma}{np}\right]^{nph}}{\Gamma(pn-p)}$

$\alpha = nph+np-1$

Just as it was true for $\lambda_o$ and $\lambda_1$, it is so for $\lambda_2$ and

$$E(\lambda_2^h) = \frac{d}{d\theta^r}h(\theta)\Bigg|_{\substack{\theta=o\\ r=-nph}}$$

Equation 4.19 is in the same form as equation 4.5, therefore

$$\frac{d}{d\theta^r}\quad h(\theta) = K_3\cdot\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\left[\frac{\sigma}{np}\right]^r$$

$$= \frac{\Gamma(nph+np-p)\left[\frac{\sigma}{np}\right]^{nph}\Gamma(nph+np-1-nph)\left[\frac{\sigma}{np}\right]^{-nph}}{\Gamma(pn-p)\quad\Gamma(nph+np-1)}$$

The $h^{th}$ moment of $\lambda_2$ is

$$E(\lambda_2^h) = \frac{\Gamma(nph+np-p)\;\Gamma(np-1)}{\Gamma(nph+np-1)\;\Gamma(np-p)}$$

The moments of the three criteria $\lambda_o$, $\lambda_1$, and $\lambda_2$ do agree with those found by P.V. Sukhatme (Ref: 18:98). However, the method used by P.V. Sukhatme requires significantly more effort.

## V. Distribution of the Criteria

In this chapter, the distributions of criteria $\lambda_o$, $\lambda_1$, and $\lambda_2$ are obtained by inverting their characteristic functions. These distributions are obtained up to an order of $n^{-4}$ and can be used to find accurate approximations to the percentage points of the test statistics.

### The Distribution of $\lambda_o$

The $h^{th}$ moment of $\lambda_o$ from Theorem 4.1 is

$$E(\lambda_o^h) = p^{nph} \frac{\Gamma(np-1)}{\Gamma(nph+np-1)} \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^p \tag{5.1}$$

Let $M=-2s \log \lambda_o$ (Def 5.1)

where 's' is to be determined later for computational ease.

Define $\phi(t)$ to be the characteristic function of M. (Def 5.2)

Then

$$\phi_m(t) = E(\lambda_o^{-2sit})$$

Since equation 5.1 holds for any complex number h, substituting -2sit for h gives

$$\phi m(t) = \frac{\Gamma(np-1)}{[\Gamma(n-1)]^p} \cdot p^{-2snpit} \cdot \frac{[\Gamma(n(1-2sit)-1)]^p}{\Gamma(np(1-2sit)-1)} \tag{5.2}$$

$$= K(n,p) \cdot Y(t) \qquad \text{(Def 5.3)}$$

where

$$K(n,p) = \frac{\Gamma(np-1)}{[\Gamma(n-1)]^p} \qquad \text{(Def 5.4)}$$

$$Y(t) = p^{-2snpit} \cdot \frac{[\Gamma(n(1-2sit)-1)]^p}{\Gamma(np(1-2sit)-1)} \qquad \text{(Def 5.6)}$$

Also define $\quad T = (1-2it)$ (Def 5.6)

The function $\phi_m(t)$ has been broken into two parts one independent of t and the second part $Y(t)$, which depends on t, shall be operated on first. Using Definition 5.6 and after some algebra

$$Y(t) = p^{-2snpit} \cdot \frac{[\Gamma(snT-sn+n-1)]^p}{\Gamma(npsT-nps+np-1)}$$

Therefore,

$$\log Y(t) = -2snpit \log p$$
$$+ p \log \Gamma(snT-sn+n-1) - \log \Gamma(npsT-nps+np-1) \qquad (5.3)$$

The expansion of several logarithmic Gamma functions will be based on the following expansion (Ref 1:204):

$$\log \Gamma(x+h) = \frac{1}{2} \log(2\pi) + (x+h-\frac{1}{2}) \log x - x$$

$$-\sum_{r=1}^{m} (-1)^r \frac{B_{r+1}(h)}{r(r+1)x^r} + R_{m+1}(x) \qquad (5.4)$$

where $R_m(x)$ is the remainder such that $R_m(x) < \theta / X^m$, $\theta$ a constant and $B_r(h)$ is the Bernoulli polynomial of degree r, order 1 defined by

$$\frac{ve^{hv}}{e^v-1} = \sum_{r=o}^{\infty} \frac{v^r B_r(h)}{r!}$$

Using the expansion of equation 5.4 on the second and third terms of equation 5.3 gives the following two expansions.

$$\log\Gamma(snT-sn+n-1) = \frac{1}{2}\log(2\pi) + (snT-sn+n-\frac{3}{2})\log(snT)$$

$$-snT \;-\sum_{r=1}^{m} \frac{(-1)^r}{r(r+1)(snT)^r} B_{r+1}(-sn+n-1) + R_{m+1} \tag{5.5}$$

$$\log\Gamma(npsT-nps+np-1) = \frac{1}{2}\log(2\pi) + (npsT-nps+np-\frac{3}{2})\log(npsT)$$

$$-npsT \;-\sum_{r=1}^{m} \frac{(-1)^r}{r(r+1)(npsT)^r} B_{r+1}(-nps+np-1) + R'_{m+1} \tag{5.6}$$

Substitute equations 5.5 and 5.6 into equation 5.3, the following results occur after some algebra.

$$\log Y(t) = (\frac{3}{2}-np)\log p + \left[\frac{p-1}{2}\right]\cdot\log(2\pi) + \frac{3(1-p)}{2}\log(snT)$$

$$+\sum_{r=1}^{m} \frac{(-1)^r}{r(r+1)(snT)^r}\left[p^{-r}B_{r+1}(-nps+np-1)-pB_{r+1}(-sn+n-1)\right]$$

$$+ R''_{m+1} \tag{5.7}$$

Define

$$C_r = \frac{(-1)^r}{r(r+1)}\left[p^{-r}B_{r+1}(-nps+np-1)-pB_{r+1}(-sn+n-1)\right] \quad \text{(Def 5.7)}$$

Thus, from equation 5.7 and Definition 5.7

$$\log Y(t) = (\frac{3}{2}-np)\log p + \left[\frac{p-1}{2}\right]\cdot\log(2\pi) + \frac{3(1-p)}{2}\log(snT)$$

$$+\sum_{r=1}^{m} \frac{C_r}{(snT)^r} + R''_{m+1} \tag{5.8}$$

Using the following relationship between $C_r$ and $D_z$ presented by Kalinin and Shalaevskii (Ref 11:6-11), $D_z$ is recursively computed as follows,

$$D_z = \sum_{k=1}^{z} k \frac{C_k D_{z-k}}{z} \quad , \quad D_o=1$$

Therefore, equation 5.8 can be transformed to

$$Y(t) = p^{3/2-np} (2\pi)^{(p-1)/2} (snT)^{3(1-p)/2} \left[ 1 + \sum_{z=1}^{\infty} \frac{D_z}{(snT)^z} \right] + R'''_{m+1} \quad (5.9)$$

From Definition 5.4,

$$K(n,p) = \frac{\Gamma(np-1)}{[\Gamma(n-1)]^p}$$

$$= \frac{\Gamma(nps-nps+np-1)}{[\Gamma(ns-ns+n-1)]^p}$$

Therefore,

$$\log K(n,p) = \log \Gamma(nps-nps+np-1) - p \log \Gamma(ns-ns+n-1) \quad (5.10)$$

By using the logarithmic expansion of equation 5.4 and some algebra, equation 5.10 becomes

$$\log K(n,p) = \left[\frac{1-p}{2}\right] \cdot \log(2\pi) + \frac{3}{2}(p-1) \cdot \log(ns) + (np-\frac{3}{2}) \cdot \log p$$

$$+ \sum_{r=1}^{m} \frac{A_r}{(ns)^r} + R'_{1,m+1}$$

$$\text{where } A_r = \frac{(-1)^r}{r(r+1)} \left[ pB_{r+1}(-ns+n-1) - p^{-r} B_{r+1}(-nps+np-1) \right] \quad (\text{Def } 5.8)$$

Once again using the Kalinin and Shalaevskii recursive relationship

$$K(n,p) = (2\pi)^{(1-p)/2} (ns)^{3(p-1)/2} p^{np-3/2} \left[ 1 + \sum_{v=1}^{\infty} \frac{B_v}{(ns)^v} \right] + R''_{1,m+1} \quad (5.11)$$

Reuniting Y(t) and K(n,p) together and some algebra results in the characteristic function of M,

$$\phi_m(t) = T^{3(1-p)/2} \left[ 1 + \sum_{v=1}^{\infty} \frac{B_v}{(ns)^v} \right] \left[ 1 + \sum_{z=1}^{\infty} \frac{D_z}{(nsT)^z} \right] + R_{2,m+1}$$

Let $\quad m=ns$ (Def 5.9)

Then up to the order $m^{-4}$, the asymptotic expansion of the distribution of $M= -2s \log \lambda_o$ is given by

$$\phi_m(t) = T^{\frac{3(1-p)}{2}} \Bigg[ 1 + m^{-1} \left[ \frac{D_1}{T} + B_1 \right] + m^{-2} \left[ \frac{D_2}{T^2} \quad \frac{B_1 D_1}{T} + B_2 \right]$$

$$+ m^{-3} \left[ \frac{D_3}{T^3} + \frac{B_1 D_2}{T^2} + \frac{B_2 D_1}{T} + B_3 \right] + m^{-4} \left[ \frac{D_4}{T^4} + \frac{B_1 D_3}{T^3} + \frac{B_2 D_2}{T^2} \right.$$

$$\left. + \frac{B_3 D_1}{T} + B_4 \right] \Bigg] + O(m^{-5}) \quad (5.12)$$

The variable 's' is chosen such that $A_1=0$. This gives

$$s = \frac{(18n-13)p-13}{18np}$$

Note from definitions 5.7 and 5.8 that $A_r = -C_r$.

Using the Kalinin and Shalaevskii recursive relation, the following equations are listed.

$$B_1 = A_1$$
$$B_2 = A_2 + 1/2\ A_1^2$$
$$B_3 = 1/3\ (A_1B_2+2A_2B_1+3A_3)$$
$$B_4 = 1/4\ (A_1B_3+2A_2B_2+3A_3B_1+4A_4)$$

Since 's' is chosen such that $A_1=0$ and noting that $A_r=-C_r$,

$$B_1 = A_1 = C_1 = D_1 = 0$$
$$B_2 = A_2 = -C_2 = -D_2$$
$$B_3 = A_3 = -C_3 = -D_3$$
$$B_4 = 1/2\ (A_2^2\ ) + A_4$$
$$D_4 = 1/2\ (A_2^2) - A_4$$

Finally, after some more algebra

$$\phi_m(t) = T^{\frac{3(1-p)}{2}} [1+m^{-2} A_2(1-T^{-2}) + m^{-3}A_3(1-T^{-3})$$
$$+ m^{-4} (A_2^2(1-T^{-2}) + (A_4-1/2(A_2^2))(1-T^{-4}))]$$
$$+ O(m^{-5}) \qquad (5.13)$$

Inverting the characteristic function in equation 5.13 and noting that $(1-2it)^{-r/2}$ is the characteristic function of a CHI Square variable with r degrees of freedom, the complementary distribution function of M is given by

$$F(x) = P(M>x) = G_{3p-3}(X) + m^{-2}A_2(G_{3p-3}(X)-G_{3p+1}(X))$$
$$+ m^{-3}A_3(G_{3p-3}(X) - G_{3p+3}(X)) + m^{-4}A_2^2(G_{3p-3}(X)-G_{3p+1}(X))$$
$$+ m^{-4}(A_4-1/2(A_2^2))(G_{3p-3}(X)-G_{3p+5}(X)) + O(m^{-5}) \qquad (5.14)$$

where $G_r(X)$ is the complementary distribution function of a CHI Square variable with r degrees of freedom. The computed value of $A_i$ (i=2,3,4) are now listed.

$$A_2 = \frac{-(47p^3 - 169p^2 + 169p - 47)}{432\ p^2}$$

$$A_3 = \frac{218p^4 - 455p^3 + 455p - 218}{14580p^3}$$

$$A_4 = \frac{-(4723p^5 - 32487p^4 + 79430p^3 - 79430p^2 + 32487p - 4723)}{466560p^4}$$

<u>The Distribution of $\lambda_1$</u>

The $h^{th}$ moment of $\lambda_1$ from Theorem 4.2 is

$$E(\lambda_1^h) = \left[\frac{\Gamma(nh+n-1)}{\Gamma(n-1)}\right]^p \cdot p^{nph} \cdot \frac{\Gamma(np-p)}{\Gamma(nph+np-p)} \tag{5.15}$$

Let $M_1 = -2u \log \lambda_1$ (Def 5.10)

where 'u' is to be computed later for computational ease.

Define $\phi_1(t)$ to be the characteristic function of $M_1$. (Def 5.11)

Then

$$\phi_1(t) = E(\lambda_1^{-2uit})$$

$$= \frac{\Gamma(np-p)}{[\Gamma(n-1)]^p}\, p^{-2npuit}\, \frac{[\Gamma(n(1-2uit)-1)]^p}{\Gamma(np(1-2uit)-p)} \tag{5.16}$$

$$= K_1(n,p)\cdot Y_1(t) \qquad \text{(Def 5.11)}$$

where

$$K_1(n,p) = \frac{\Gamma(np-p)}{[\Gamma(n-1)]^p} \qquad \text{(Def 5.12)}$$

$$Y_1(t) = p^{-2npuit} \frac{[\Gamma(n(1-2uit)-1)]^p}{\Gamma(np(1-2uit)-p)} \qquad \text{(Def 5.13)}$$

As before $\phi_1(t)$ is broken into two parts to be operated on. Using Definition 5.6 and after some algebra

$$Y_1(t) = p^{-2npuit} \frac{[\Gamma(unT+n-un-1)]^p}{\Gamma(nupT+np-nup-p)}$$

Therefore,

$$\log Y_1(t) = -2npuit \log p + p \log \Gamma(unT+n-un-1) - \log \Gamma(nupT+np-nup-p) \qquad (5.17)$$

Using the expansion of equation 5.4 on the second and third terms of equation 5.17, the following results occur after some algebra.

$$\log Y_1(t) = (-pn+p+1/2) \log p + 1/2(p-1) \log (2\pi) + 1/2(1-p) \log (unT) + \sum_{r=1}^{m} \frac{C_r}{(unT)^r} + R_{m+1} \qquad (5.18)$$

where

$$C_r = \frac{(-1)^r}{r(r+1)} [p^{-r}B_{r+1}(np-npu-p)-pB_{r+1}(n-nu-1)] \qquad \text{(Def 5.14)}$$

Once again using the Kalinin and Shalaevskii recursive relationship equation 5.18 can be transformed to

$$Y_1(t) = p^{-pn+p+1/2}(2\pi)^{1/2(p-1)}(unT)^{1/2(1-p)}\left[1+\sum_{s=1}^{\infty}\frac{D_s}{(unT)^s}\right]+R'_{m+1} \tag{5.19}$$

From Definition 5.12,

$$K_1(n,p) = \frac{\Gamma(np-p)}{[\Gamma(n-1)]^p}$$

$$= \frac{\Gamma(nup+np-nup-p)}{[\Gamma(un+n-un-1)]^p}$$

Therefore,

$$\log K_1(n,p) = \log \Gamma(nup+np-nup-p) - p \log \Gamma(un+n-un-1) \tag{5.20}$$

Using the logarithmic expansion of equation 5.4 and some algebra, equation 5.20 becomes

$$\log K_1(n,p) = \frac{1}{2}(1-p)\log(2\pi) + \frac{1}{2}(p-1)\log(un) + (np-p-\frac{1}{2})\log p$$

$$+\sum_{r=1}^{m}\frac{A_r}{(un)^r} + R_{1,m+1}$$

where

$$A_r = \frac{(-1)^r}{r(r+1)}\left[pB_{r+1}(n-un-1)-p^{-r}B_{r+1}(np-nup-p)\right] \qquad \text{(Def 5.15)}$$

Using the Kalinin and Shalaevskii recursive relationship

$$K_1(n,p) = (2\pi)^{(1-p)/2}(un)^{(p-1)/2}p^{np-p-1/2}\left[1+\sum_{v=1}^{\infty}\frac{B_v}{(un)^v}\right] + R'_{1,m+1} \quad (5.21)$$

Reuniting $Y_1(t)$ and $K_1(n,p)$ and some algebra results in the characteristic function of $M_1$.

$$\phi_1(t) = T^{\frac{(1-p)}{2}}\left[1+\sum_{v=1}^{\infty}\frac{B_v}{(un)^v}\right]\left[1+\sum_{s=1}^{\infty}\frac{D_s}{(unT)^s}\right]+R_{2,m+1}$$

Let $\quad w=un$ (Def 5.16)

Then up to the order $w^{-4}$, the asymptotic expansion of the distribution $M_1 = -2u\log\lambda_1$ is given by

$$\phi_1(t) = T^{\frac{(1-p)}{2}}\left[1+w^{-1}\left[\frac{D_1}{T}+B_1\right]+w^{-2}\left[\frac{D_2}{T^2}\ \frac{B_1D_1}{T}+B_2\right]\right.$$
$$+w^{-3}\left[\frac{D_3}{T^3}+\frac{B_1D_2}{T^2}+\frac{B_2D_1}{T}+B_3\right]+w^{-4}\left[\frac{D_4}{T^4}+\frac{B_1D_3}{T^3}\right.$$
$$\left.\left.+\frac{B_2D_2}{T^2}+\frac{B_3D_1}{T}+B_4\right]\right]+O(w^{-5}) \quad (5.22)$$

The variable 'u' is now chosen such that $A_1=0$. This gives

$$u = \frac{(6n-7)p-1}{6np}$$

Note from Definitions 5.14 and 5.15 that $A_r = -C_r$. Using the previous listed Kalinin and Shalaevskii variable relationships and some algebra

$$\phi_1(t) = T^{\frac{(1-p)}{2}} [1+w^{-2} A_2(1-T^{-2}) + w^{-3}A_3(1-T^{-3})$$
$$+ w^{-4} (A_2{}^2(1-T^{-2}) + (A_4-1/2(A_2{}^2))(1-T^{-4}))]$$
$$+ O(w^{-5}) \qquad (5.23)$$

Inverting the characteristic function in equation 5.23, the complementary distribution function of $M_1$ is given by

$$F(x) = P(M>x) = G_{p-1}(X) + w^{-2}A_2(G_{p-1}(X)-G_{p+3}(X))$$
$$+ w^{-3}A_3(G_{p-1}(X) - G_{p+5}(X)) + w^{-4}A_2{}^2(G_{p-1}(X)-G_{p+3}(X))$$
$$+ w^{-4}(A_4-1/2(A_2{}^2))(G_{p-1}(X)-G_{p+7}(X)) + O(w^{-5}) \qquad (5.24)$$

where $G_r(X)$ is the complementary distribution function of a CHI Square variable with r degrees of freedom. The computed values of $A_i$ (i=2,3,4) are now listed.

$$A_2 = \frac{p^3 + p^2 - p - 1}{144\ p^2}$$

$$A_3 = \frac{2p^4 - 5p^3 + 5p - 2}{1620p^3}$$

$$A_4 = \frac{-(19p^5 + 9p^4 - 10p^3 + 10p^2 - 9p - 19}{17280\ p^4}$$

The Distribution of $\lambda_2$

The $h^{th}$ moment of $\lambda_2$ from Theorem 4.3 is

$$E(\lambda_2^h) = \frac{\Gamma(nph + np - p)\;\;\Gamma(np - 1)}{\Gamma(nph + np - 1)\;\;\Gamma(np - p)} \qquad (5.25)$$

Let $M_2 = -2b \log \lambda_2$ (Def 5.17)

where 'b' is to be computed later for computational ease.

Define $\phi_2(t)$ to be the characteristic function of $M_2$. (Def 5.18)

Then

$$\phi_2(t) = E(\lambda_2^{-2bit})$$

$$= \frac{\Gamma(np-1)\Gamma(np(1-2bit)-p)}{\Gamma(np-p)\Gamma(np(1-2bit)-1)} \qquad (5.26)$$

$$= K_2(n,p)\cdot Y_2(t) \qquad \text{(Def 5.19)}$$

where

$$K_2(n,p) = \frac{\Gamma(np-1)}{\Gamma(np-p)} \qquad \text{(Def 5.20)}$$

$$Y_2(t) = \frac{\Gamma(np(1-2bit)-p)}{\Gamma(np(1-2bit)-1)} \qquad \text{(Def 5.21)}$$

As before $\phi_2(t)$ is broken into two parts to be operated on.

Using Definition 5.6 and after some algebra

$$Y_2(t) = \frac{\Gamma(npbT+np-npb-p)}{\Gamma(npbT+np-npb-1)}$$

Therefore,

$$\log Y_2(t) = \log \Gamma(npbT+np-npb-p) - \log \Gamma(npbT+np-npb-1) \qquad (5.27)$$

Using the expansion of equation 5.4 on the two terms of equation 5.27, the following results occur after some algebra.

$$\log Y_2(t) = (1-p)\log(npbT) + \sum_{r=1}^{m} \frac{C_r}{(npbT)^r} + R_{m+1} \tag{5.28}$$

where

$$C_r = \frac{(-1)^r}{r(r+1)}\left[B_{r+1}(np(1-b)-1) - B_{r+1}(p(n-nb-1))\right] \qquad \text{(Def 5.22)}$$

Once again using the Kalinin and Shalaevskii recursive relationship equation 5.28 can be transformed to

$$Y_2(t) = (npbT)^{1-p}\left[1 + \sum_{s=1}^{\infty} \frac{D_s}{(npbT)^s}\right] + R'_{m+1} \tag{5.29}$$

From Definition 5.20

$$K_2(n,p) = \frac{\Gamma(np-1)}{\Gamma(np-p)}$$

$$= \frac{\Gamma(npb+np(1-b)-1)}{\Gamma(npb+p(n-nb-1))}$$

Therefore,

$$\log K_2(n,p) = \log\Gamma(npb+np(1-b)-1) - \log\Gamma(npb+p(n-nb-1)) \tag{5.30}$$

Using the logarithmic expansion of equation 5.4 and some algebra, equation 5.30 becomes

$$\log K_2(n,p) = (p-1)\log(npb) + \sum_{r=1}^{m} \frac{A_r}{(npb)^r} + R_{1,m+1}$$

where

$$A_r = \frac{(-1)}{r(r+1)} \left[B_{r+1}(p(n-nb-1)) - B_{r+1}(np(1-b)-1)\right] \qquad \text{(Def 5.23)}$$

Using the Kalinin and Shalaevskii recursive relationship

$$K_2(n,p) = (npb)^{p-1} \left[ 1 + \sum_{v=1}^{\infty} \frac{B_v}{(npb)^v} \right] + R'_{1,m+1} \qquad (5.31)$$

Reuniting $Y_2(t)$ and $K_2(n,p)$ and some algebra results in the characteristic function of $M_2$.

$$\phi_2(t) = T^{1-p} \left[1 + \sum_{v=1}^{\infty} \frac{B_v}{(npb)^v}\right]\left[1 + \sum_{s=1}^{\infty} \frac{D_s}{(npbT)^s}\right] + R_{2,m+1}$$

Let $\quad z = npb$ (Def 5.24)

Then up to the order $z^{-4}$, the asymptotic expansion of the distribution $M_2 = -2b \log \lambda_2$ is given by

$$\phi_2(t) = T^{1-p} \left[1 + z^{-1}\left[\frac{D_1}{T} + B_1\right] + z^{-2}\left[\frac{D_2}{T^2} \quad \frac{B_1 D_1}{T} + B_2\right] + z^{-3}\left[\frac{D_3}{T^3} + \frac{B_1 D_2}{T^2} + \frac{B_2 D_1}{T} + B_3\right] + z^{-4}\left[\frac{D_4}{T^4} + \frac{B_1 D_3}{T^3} + \frac{B_2 D_2}{T^2} + \frac{B_3 D_1}{T} + B_4\right]\right] + O(z^{-5}) \qquad (5.32)$$

The variable 'b' is now chosen such that $A_1=0$. This gives

$$b = \frac{(2n-1)p-2}{2np}$$

Note from Definitions 5.22 and 5.23 that $A_r = -C_r$.

Using the previously listed Kalinin and Shalaevskii variable relationships and some algebra

$$\phi_2(t) = T^{1-p}\,[1+z^{-2}\,A_2(1-T^{-2}) + z^{-3}A_3(1-T^{-3})$$
$$+ z^{-4}\,(A_2{}^2(1-T^{-2}) + (A_4-1/2(A_2{}^2))(1-T^{-4}))]$$
$$+ O(z^{-5}) \qquad (5.33)$$

Inverting the characteristic function in equation 5.33, the complementary distribution function of $M_2$ is given by

$$F(x) = P(M_2>x) = G_{2p-2}(X) + z^{-2}A_2(G_{2p-2}(X)-G_{2p+2}(X))$$
$$+ z^{-3}A_3(G_{2p-2}(X) - G_{2p+4}(X)) + z^{-4}A_2{}^2(G_{2p-2}(X)-G_{2p+2}(X))$$
$$+ z^{-4}(A_4-1/2(A_2{}^2))(G_{2p-2}(X)-G_{2p+6}(X)) + O(z^{-5}) \qquad (5.34)$$

where $G_r(X)$ is the complementary distribution function of a CHI Square variable with r degrees of freedom. The computed values of $A_i$ (i=2,3,4) are now listed.

$$A_2 = \frac{-(p^3 - 3p^2 + 2p)}{24}$$

$$A_3 = 0$$

$$A_4 = \frac{-(3p^5 - 15p^4 + 20p^3 - 8p)}{960}$$

Computation of Percentage Points

The percentage points of $L_i$ and $M_i$ $(i=0,1,2)$ have been computed using the Newton-Secant method for finding zeros of a function. The arithmetic is carried using Double Precision on a CDC Cyber 750 system at Wright-Patterson AFB, OH. A subroutine of International Mathematical and Statistical Libraries (IMSL) was used in finding the percentage points. The computation of the various coefficients $A_i$ was done using the MACSYMA system of Massachusetts Institute of Technology. The percentage points are computed for $p = 2(1)10$, $n = 10(1)20(5)50(10)100$, and $\alpha = .100, .050, .025, .010, .005$.

A good approximation to the asymptotic expansion of the test statistic distributions is the first term of the expansion. This approximation is accurate to three significant figures as a result of the computational technique of solving variables s, u, b so that $A_1 = 0$ and the characteristics of the expansions. The first term of the expansion is a CHI-Square distribution and therefore its' percentage points are readily available. To use this approximation the correct percentage points for $M_0$ are found by looking up a CHI-Square percentage point for the appropriate alpha with 3p-3 degrees of freedom, for $M_1$ and $M_2$ percentage points are found by using p-1 and 2p-2 degrees of freedom respectively.

## VI. Test Statistics and Practical Illustrations

In this chapter the tests considered in this thesis will be applied to a set of aircraft life data. First a test is performed to determine if the underlying distribution of the samples is exponentially distributed. The procedure chosen is the Bartlett Test. After checking to assure that the distributions are exponential, the three hypotheses stated in Chapter I are tested using the test statistics $L_i$ and $M_i$ $(i=0,1,2)$. The data is a sample of the intervals between successive failures (given in operating hours) of airconditioning equipment on four Boeing 720 aircraft (Ref 17:376). The data is given in Table 6.1.

TABLE 6.1

INTERVAL BETWEEN FAILURES OF AIRCRAFT AIRCONDITIONING

| 7909 (1) | 7912 (2) | 7913 (3) | 7914 (4) |
|---|---|---|---|
| 90 | 23 | 97 | 50 |
| 10 | 261 | 51 | 44 |
| 60 | 87 | 11 | 102 |
| 186 | 7 | 4 | 72 |
| 61 | 120 | 141 | 22 |
| 49 | 14 | 18 | 39 |
| 14 | 62 | 142 | 3 |
| 24 | 47 | 68 | 15 |
| 56 | 225 | 77 | 197 |
| 20 | 71 | 80 | 188 |
| 79 | 246 | 1 | 79 |
| 84 | 21 | 16 | 88 |
| 44 | 42 | 106 | 46 |
| 59 | 20 | 206 | 5 |
| 29 | 5 | 82 | 5 |
| 118 | 12 | 54 | 36 |
| 25 | 120 | 31 | 22 |
| 156 | 11 | 216 | 139 |
| 310 | 3 | 46 | 210 |
| 76 | 14 | 111 | 97 |
| 26 | 71 | 39 | 30 |
| 44 | 11 | 63 | 23 |
| 23 | 14 | 18 | 13 |
| 62 | 11 | 191 | 14 |

Bartlett's Test is used to determine if each of the samples failure intervals are exponentially distributed. The Bartlett Test statistic (Ref 12: 238,239) is

$$B_r = \frac{2r\left[\log\left[\frac{t_r}{r}\right] - \frac{1}{r}\left[\sum_{i=1}^{r} \log X_i\right]\right]}{1 + ((r+1)/6r)}$$

where $X_i$ is time to failure (interval), r is number of failures and $t_r = \sum_{i=1}^{r} X_i$. The statistic $B_r$ is CHI-Square distributed with r-1 degrees of freedom when the sample being tested has an underlying exponential distribution. A two-tailed CHI-Square test is used for this test. The statistics $B_{24}$ for each aircraft are given in Table 6.2.

TABLE 6.2

$B_{24}$ STATISTICS

| Airplane | 7909 (1) | 7912 (2) | 7913 (3) | 7914 (4) |
|---|---|---|---|---|
| $B_{24}$ | 13.487 | 29.546 | 20.510 | 22.180 |

In order for these four samples to have an underlying exponential distribution with a 95% confidence interval they must fall between 11.688 and 38.076. All four statistics are within that region and are therefore exponentially distributed. One can also use the procedures given in Epstein (Ref 3) for testing whether the four populations are exponentially distributed.

Test Statistics for $H_o$

To determine if the airconditioners are failing after the same interval or they have identical location and scale parameters, the first hypothesis $H_o$ is used. $H_o$ states

$$\theta_1 = \theta_2 = \ldots = \theta_p \text{ and } \sigma_1 = \sigma_2 = \ldots = \sigma_p$$

against the alternative that one of the parameters is not equal.

A. Let $L_o = \lambda_o^{\frac{1}{np}}$

$$= \frac{\left[\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})\right]^{\frac{1}{p}}}{\bar{X} - X_{(1)}} \qquad \text{(Def 6.1)}$$

where $\bar{X}_i$ is the $i^{th}$ sample mean, $X_{i(1)}$ is the minimum observation in the $i^{th}$ sample, $\bar{X}$ is the grand mean of all observations, $X_{(1)}$ is the minimum observation of all the observations and p is the number of populations being tested. These definitions will be used for all test statistics.

B. $M_o = -2s \log \lambda_o$ from Definition 5.1

Where

$$s = \frac{(18n - 13)p - 13}{18np}$$

$$\lambda_o = \frac{\prod_{i=1}^{p} (\bar{X}_i - X_{i(1)})^n}{(\bar{X}-X_{(1)})^{np}}$$

Decision Rule for $H_o$: $H_o$ is rejected if $L_o < L_T$ or if $M_o > M_T$, where $L_T$ and $M_T$ are values obtained from Tables in the Appendix corresponding to $L_o$ and $M_o$. $L_o$ and $M_o$ are equivalent test statistics.

In the example of aircraft airconditioning equipment, the calculations for $L_o$ are now shown.

$\overline{X}_1 = 71.0417$ $\overline{X}_2 = 63.250$ $\overline{X}_3 = 77.875$ $\overline{X}_4 = 64.125$

$X_{1(1)} = 10$ $X_{2(1)} = 3$ $X_{3(1)} = 1$ $X_{4(1)} = 3$

$\overline{X} = 69.0729$ $X_{(1)} = 1$ $p = 4$

$$L_o = \frac{[(61.0417)(60.250)(76.875)(61.125)]^{1/4}}{68.0729} = \frac{64.475818}{68.0729}$$

$$= .947158$$

From Table A3 after interpolating

$L_{T,\alpha=.025} = .901399$

Since $L_o > L_T$, by the decision rule, accept the null hypothesis. For this example $M_o = 10.0314$

$M_{T,\alpha=.025} = 19.0257$

The same result is obtained, since $M_o < M_T$, by the decision rule, accept the null hypothesis.

## Test Statistics for $H_1$

If the question is to determine if the scale parameters (mean/variance) are equal and the location parameter is not significant, the second hypothesis $H_1$ should be used. $H_1$ states $\sigma_1 = \sigma_2 = \ldots = \sigma_p$ and $\theta_i$'s are unspecified $(i=1,2,\ldots,p)$ against alternative that one of the scale parameters is not equal to the others.

A. Let $L_1 = \lambda_1^{\frac{1}{np}}$

$$= \frac{\left[\prod_{i=1}^{p} (\overline{X}_i - X_{i(1)})\right]^{\frac{1}{p}}}{\frac{1}{p} \sum_{i=1}^{p} (\overline{X}_i - X_{i(1)})} \qquad \text{(Def 6.2)}$$

B. $M_1 = -2u \log \lambda_1$ from Definition 5.10

Where

$$u = \frac{(6n-7)p-1}{6np}$$

$$\lambda_1 = \frac{\prod_{i=1}^{p} (\overline{X}_i - X_{i(1)})^n}{\left[\frac{1}{p} \sum_{i=1}^{p} (\overline{X}_i - X_{i(1)})\right]^{np}}$$

Decision Rule for $H_1$: $H_1$ is rejected if $L_1 < L_T$ or if $M_1 > M_T$, where $L_T$ and $M_T$ are values obtained from Tables in the Appendix corresponding to $L_1$ and $M_1$. $L_1$ and $M_1$ are equivalent test statistics.

In the example of aircraft airconditioning equipment, $L_1$ is shown.

$$L_1 = \frac{[(61.0417)(60.250)(76.875)(61.125)]^{1/4}}{1/4\ (61.0417 + 60.250 + 76.875 + 61.125)} = \frac{64.475818}{64.822925}$$

$$= .994645$$

From Table A12 after interpolating

$$L_{T,\alpha=.025} = .949605$$

Since $L_1 > L_T$, by the decision rule, accept the null hypothesis. For this example $M_1 = .978963$

$$M_{T,\alpha=.025} = 9.34725$$

The same result is obtained, since $M_1 < M_T$, by the decision rule accept the null hypothesis.

<u>Test Statistics for $H_2$</u>

The third hypothesis $H_2$ is used when it is known that all scale parameters (mean/variance) are equal and testing to determine if the location parameters (guaranteed life/minimum failure time) are equal. $H_2$ states

$$\theta_1 = \theta_2 = \dots = \theta_p, \text{ given that } \sigma_1 = \sigma_2 = \dots = \sigma_p$$

against alternative that one of the location parameters is not equal to the others.

A. Let $L_2 = \lambda_2^{\frac{1}{np}}$

$$= \frac{\frac{1}{p}\sum_{i=1}^{p}(\bar{X}_i - X_{i(1)})}{\bar{\bar{X}} - X_{(1)}} \qquad \text{(Def 6.3)}$$

B. $M_2 = -2b \log \lambda_2$ from Definition 5.17.

Where

$$b = \frac{(2n-1)p-2}{2np}$$

$$\lambda_2 = \frac{\left[\frac{1}{p}\sum_{i=1}^{p}(\bar{X}_i - X_{i(1)})\right]^{np}}{\left[\bar{X} - X_{(1)}\right]^{np}}$$

Decision Rule for $H_2$: $H_2$ is rejected if $L_2$ $L_T$ or if $M_2 > M_T$, where $L_T$ and $M_T$ are values obtained from Tables in the Appendix corresponding to $L_2$ and $M_2$. $L_2$ and $M_2$ are equivalent test statistics.

In the example of aircraft airconditioning equipment, $L_2$ is now shown.

$$L_2 = \frac{1/4\ (61.0417 + 60.250 + 76.875 + 61.125)}{68.0729} = \frac{64.822925}{68.0729}$$

$$= .952257$$

From Table A21 after interpolating

$$L_{T,\ =.025} = .924660$$

Since $L_2 > L_T$, by the decision rule, accept the null hypothesis that all of the location parameters are equal. For this example $M_2 = 9.09918$

$$M_{T,\ =.025} = 14.45098$$

The same result is obtained, since $M_2 < M_T$, by the decision rule, accept the null hypothesis.

## VII. Conclusion

In this thesis three hypotheses connected with the testing of equality of several exponential populations are considered. The asymptotic expansions of the distributions are found based on the Neyman-Pearson likelihood ratio criteria for $\lambda_0$, $\lambda_1$, and $\lambda_2$. Tables of percentage points are computed from these expansions for each test statistic distribution using the Newton-Secant method for finding zeros of a function. The CHI-Square distribution with the appropriate degrees of freedom is found to be a good approximation to the percentage points for $M_0$, $M_1$, and $M_2$. A practical illustration in Chapter VI combines definitions, test statistics, and percentage point tables to make this thesis a product that can be applied to many situations in both the Air Force and Industry.

The situations where these tests can be applied include the time interval between failure of parts on aircraft, time to failure of identical electronic components, the interval between consecutive accidents incurred by the same individual, and other life testing experiments involving the exponential distribution.

In order to make these tests more useful to the analyst further study should be continued in developing tests and tables for testing samples with unequal sizes. Since the test statistics are based on the likelihood ratio test it is known that the tests are uniformly most powerful. In order to find the power of the test statistics, further

research should be conducted to find the non-central distributions of the criteria. This additional research would enhance further applications of tests of hypothesis connected with exponential populations.

# APPENDIX

TABLES

Table A1 Percentage Points of $L_o$ and $M_o$=-2s log $\lambda_o$

P = 2

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 6.25069 | 7.81371 | 9.34702 | 11.34292 | 12.83572 |
| | 11 | 6.25083 | 7.81392 | 9.34730 | 11.34332 | 12.83622 |
| | 12 | 6.25093 | 7.81407 | 9.34751 | 11.34361 | 12.83658 |
| | 13 | 6.25101 | 7.81418 | 9.34766 | 11.34382 | 12.83685 |
| | 14 | 6.25107 | 7.81426 | 9.34778 | 11.34399 | 12.83706 |
| | 15 | 6.25111 | 7.81433 | 9.34787 | 11.34411 | 12.83722 |
| | 16 | 6.25115 | 7.81439 | 9.34794 | 11.34422 | 12.83735 |
| | 17 | 6.25118 | 7.81443 | 9.34800 | 11.34430 | 12.83745 |
| | 18 | 6.25121 | 7.81446 | 9.34805 | 11.34437 | 12.83754 |
| | 19 | 6.25123 | 7.81449 | 9.34809 | 11.34442 | 12.83761 |
| | 20 | 6.25124 | 7.81452 | 9.34812 | 11.34447 | 12.83767 |
| | 25 | 6.25130 | 7.81460 | 9.34823 | 11.34462 | 12.83786 |
| | 30 | 6.25133 | 7.81464 | 9.34829 | 11.34470 | 12.83795 |
| | 35 | 6.25135 | 7.81467 | 9.34832 | 11.34475 | 12.83801 |
| | 40 | 6.25136 | 7.81468 | 9.34834 | 11.34478 | 12.83805 |
| | 45 | 6.25136 | 7.81469 | 9.34835 | 11.34480 | 12.83807 |
| | 50 | 6.25137 | 7.81470 | 9.34836 | 11.34481 | 12.83809 |
| | 60 | 6.25137 | 7.81471 | 9.34838 | 11.34483 | 12.83811 |
| | 70 | 6.25138 | 7.81471 | 9.34838 | 11.34484 | 12.83812 |
| | 80 | 6.25138 | 7.81472 | 9.34839 | 11.34485 | 12.83813 |
| | 90 | 6.25138 | 7.81472 | 9.34839 | 11.34485 | 12.83814 |
| | 100 | 6.25138 | 7.81472 | 9.34839 | 11.34485 | 12.83814 |
| $L_o$ | 10 | .839245 | .803261 | .769460 | .727584 | .697760 |
| | 11 | .854205 | .821199 | .790060 | .751288 | .723538 |
| | 12 | .866623 | .836149 | .807296 | .771223 | .745301 |
| | 13 | .877095 | .848798 | .821926 | .788217 | .763911 |
| | 14 | .886044 | .859638 | .834498 | .802871 | .780002 |
| | 15 | .893780 | .869030 | .845417 | .815637 | .794050 |
| | 16 | .900534 | .877246 | .854987 | .826855 | .806420 |
| | 17 | .906480 | .884494 | .863443 | .836790 | .817393 |
| | 18 | .911757 | .890934 | .870969 | .845649 | .827193 |
| | 19 | .916470 | .896695 | .877711 | .853599 | .835998 |
| | 20 | .920705 | .901878 | .883783 | .860771 | .843951 |
| | 25 | .936745 | .921562 | .906906 | .888176 | .874421 |
| | 30 | .947389 | .934670 | .922359 | .906576 | .894947 |
| | 35 | .954967 | .944026 | .933414 | .919779 | .909710 |
| | 40 | .960637 | .951038 | .941714 | .929714 | .920838 |
| | 45 | .965039 | .956489 | .948175 | .937460 | .929525 |
| | 50 | .968556 | .960848 | .953346 | .943669 | .936494 |
| | 60 | .973822 | .967384 | .961109 | .953001 | .946982 |
| | 70 | .977578 | .972050 | .966657 | .959681 | .954497 |
| | 80 | .980391 | .975548 | .970820 | .964699 | .960146 |
| | 90 | .982577 | .978268 | .974058 | .968606 | .964548 |
| | 100 | .984325 | .980443 | .976650 | .971734 | .968074 |

Table A2 Percentage Points of $L_o$ and $M_o = -2s \log \lambda_o$

P = 3

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 10.64976 | 12.59824 | 14.45765 | 16.82245 | 18.55997 |
| | 11 | 10.64882 | 12.59701 | 14.45613 | 16.82052 | 18.55771 |
| | 12 | 10.64811 | 12.59610 | 14.45499 | 16.81907 | 18.55602 |
| | 13 | 10.64757 | 12.59540 | 14.45412 | 16.81796 | 18.55471 |
| | 14 | 10.64715 | 12.59485 | 14.45344 | 16.81709 | 18.55369 |
| | 15 | 10.64681 | 12.59441 | 14.45289 | 16.81639 | 18.55287 |
| | 16 | 10.64653 | 12.59405 | 14.45245 | 16.81583 | 18.55221 |
| | 17 | 10.64631 | 12.59376 | 14.45208 | 16.81536 | 18.55166 |
| | 18 | 10.64612 | 12.59352 | 14.45178 | 16.81497 | 18.55121 |
| | 19 | 10.64596 | 12.59331 | 14.45153 | 16.81465 | 18.55083 |
| | 20 | 10.64583 | 12.59314 | 14.45131 | 16.81437 | 18.55050 |
| | 25 | 10.64539 | 12.59257 | 14.45060 | 16.81346 | 18.54943 |
| | 30 | 10.64516 | 12.59226 | 14.45022 | 16.81297 | 18.54885 |
| | 35 | 10.64502 | 12.59208 | 14.44999 | 16.81268 | 18.54851 |
| | 40 | 10.64493 | 12.59196 | 14.44984 | 16.81249 | 18.54829 |
| | 45 | 10.64487 | 12.59188 | 14.44974 | 16.81237 | 18.54814 |
| | 50 | 10.64482 | 12.59183 | 14.44967 | 16.81228 | 18.54803 |
| | 60 | 10.64477 | 12.59175 | 14.44958 | 16.81216 | 18.54790 |
| | 70 | 10.64473 | 12.59171 | 14.44953 | 16.81209 | 18.54781 |
| | 80 | 10.64471 | 12.59168 | 14.44949 | 16.81204 | 18.54776 |
| | 90 | 10.64470 | 12.59166 | 14.44947 | 16.81201 | 18.54772 |
| | 100 | 10.64469 | 12.59165 | 14.44945 | 16.81199 | 18.54770 |
| $L_o$ | 10 | .821676 | .792673 | .765951 | .733264 | .710139 |
| | 11 | .837926 | .811253 | .786591 | .756307 | .734802 |
| | 12 | .851468 | .826786 | .803900 | .775708 | .755626 |
| | 13 | .862925 | .839963 | .818621 | .792261 | .773437 |
| | 14 | .872743 | .851281 | .831292 | .806549 | .788840 |
| | 15 | .881251 | .861106 | .842313 | .819004 | .802290 |
| | 16 | .888693 | .869715 | .851984 | .829956 | .814136 |
| | 17 | .895258 | .877321 | .860540 | .839662 | .824647 |
| | 18 | .901093 | .884089 | .868163 | .848323 | .834036 |
| | 19 | .906312 | .890150 | .874996 | .856097 | .842473 |
| | 20 | .911009 | .895609 | .881157 | .863115 | .850095 |
| | 25 | .928846 | .916390 | .904659 | .889959 | .879311 |
| | 30 | .940728 | .930273 | .920405 | .908007 | .899005 |
| | 35 | .949211 | .940204 | .931689 | .920972 | .913177 |
| | 40 | .955569 | .947659 | .940172 | .930736 | .923864 |
| | 45 | .960513 | .953461 | .946781 | .938353 | .932208 |
| | 50 | .964467 | .958106 | .952075 | .944461 | .938905 |
| | 60 | .970396 | .965077 | .960028 | .953647 | .948985 |
| | 70 | .974629 | .970059 | .965718 | .960226 | .956210 |
| | 80 | .977803 | .973797 | .969990 | .965169 | .961643 |
| | 90 | .980272 | .976706 | .973315 | .969020 | .965877 |
| | 100 | .982246 | .979033 | .975977 | .972104 | .969269 |

Table A3 Percentage Points of $L_o$ and $M_o = -2s \log \lambda_o$

P = 4

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 14.69559 | 16.93387 | 19.04070 | 21.68812 | 23.61477 |
| | 11 | 14.69338 | 16.93112 | 19.03740 | 21.68405 | 23.61010 |
| | 12 | 14.69173 | 16.92906 | 19.03493 | 21.68101 | 23.60661 |
| | 13 | 14.69047 | 16.92749 | 19.03303 | 21.67867 | 23.60393 |
| | 14 | 14.68949 | 16.92626 | 19.03155 | 21.67684 | 23.60183 |
| | 15 | 14.68870 | 16.92528 | 19.03037 | 21.67538 | 23.60015 |
| | 16 | 14.68806 | 16.92448 | 19.02941 | 21.67420 | 23.59879 |
| | 17 | 14.68754 | 16.92383 | 19.02862 | 21.67323 | 23.59767 |
| | 18 | 14.68710 | 16.92328 | 19.02796 | 21.67242 | 23.59674 |
| | 19 | 14.68673 | 16.92282 | 19.02741 | 21.67173 | 23.59596 |
| | 20 | 14.68642 | 16.92244 | 19.02694 | 21.67116 | 23.59529 |
| | 25 | 14.68540 | 16.92116 | 19.02540 | 21.66925 | 23.59310 |
| | 30 | 14.68486 | 16.92048 | 19.02458 | 21.66824 | 23.59193 |
| | 35 | 14.68453 | 16.92007 | 19.02409 | 21.66763 | 23.59124 |
| | 40 | 14.68432 | 16.91981 | 19.02378 | 21.66724 | 23.59079 |
| | 45 | 14.68418 | 16.91963 | 19.02356 | 21.66698 | 23.59048 |
| | 50 | 14.68408 | 16.91951 | 19.02341 | 21.66679 | 23.59026 |
| | 60 | 14.68395 | 16.91934 | 19.02321 | 21.66654 | 23.58998 |
| | 70 | 14.68387 | 16.91925 | 19.02309 | 21.66640 | 23.58981 |
| | 80 | 14.68382 | 16.91918 | 19.02302 | 21.66630 | 23.58970 |
| | 90 | 14.68379 | 16.91914 | 19.02296 | 21.66624 | 23.58963 |
| | 100 | 14.68376 | 16.91911 | 19.02293 | 21.66619 | 23.58958 |
| $L_o$ | 10 | .817157 | .792408 | .769797 | .742298 | .722905 |
| | 11 | .833686 | .810908 | .790037 | .764571 | .746556 |
| | 12 | .847479 | .826389 | .807017 | .783318 | .766511 |
| | 13 | .859164 | .839532 | .821464 | .799311 | .783567 |
| | 14 | .869187 | .850829 | .833904 | .813114 | .798312 |
| | 15 | .877881 | .860642 | .844727 | .825146 | .811182 |
| | 16 | .885492 | .869246 | .854228 | .835725 | .822514 |
| | 17 | .892210 | .876850 | .862635 | .845101 | .832566 |
| | 18 | .898185 | .883620 | .870127 | .853466 | .841543 |
| | 19 | .903532 | .889685 | .876845 | .860975 | .849608 |
| | 20 | .908347 | .895149 | .882903 | .867754 | .856893 |
| | 25 | .926651 | .915967 | .906023 | .893682 | .884808 |
| | 30 | .938863 | .929890 | .921522 | .911116 | .903617 |
| | 35 | .947590 | .939856 | .932634 | .923640 | .917149 |
| | 40 | .954137 | .947342 | .940991 | .933071 | .927351 |
| | 45 | .959230 | .953171 | .947503 | .940430 | .935316 |
| | 50 | .963305 | .957838 | .952721 | .946331 | .941708 |
| | 60 | .969419 | .964846 | .960562 | .955206 | .951328 |
| | 70 | .973786 | .969856 | .966172 | .961563 | .958223 |
| | 80 | .977062 | .973616 | .970385 | .966340 | .963407 |
| | 90 | .979610 | .976543 | .973665 | .970060 | .967446 |
| | 100 | .981649 | .978885 | .976290 | .973041 | .970688 |

Table A4 Percentage Points of $L_o$ and $M_o = -2s \log \lambda_o$

P = 5

| | N | $\alpha$=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 18.56820 | 21.04900 | 23.36372 | 26.24959 | 28.33646 |
| | 11 | 18.56471 | 21.04477 | 23.35873 | 26.24358 | 28.32966 |
| | 12 | 18.56211 | 21.04161 | 23.35500 | 26.23910 | 28.32459 |
| | 13 | 18.56012 | 21.03918 | 23.35215 | 26.23565 | 28.32069 |
| | 14 | 18.55856 | 21.03729 | 23.34991 | 26.23296 | 28.31764 |
| | 15 | 18.55732 | 21.03577 | 23.34813 | 26.23081 | 28.31520 |
| | 16 | 18.55631 | 21.03455 | 23.34668 | 26.22906 | 28.31323 |
| | 17 | 18.55548 | 21.03354 | 23.34549 | 26.22763 | 28.31160 |
| | 18 | 18.55479 | 21.03270 | 23.34450 | 26.22643 | 28.31025 |
| | 19 | 18.55421 | 21.03200 | 23.34367 | 26.22543 | 28.30911 |
| | 20 | 18.55372 | 21.03140 | 23.34296 | 26.22458 | 28.30815 |
| | 25 | 18.55211 | 21.02943 | 23.34064 | 26.22177 | 28.30497 |
| | 30 | 18.55125 | 21.02838 | 23.33940 | 26.22027 | 28.30327 |
| | 35 | 18.55073 | 21.02776 | 23.33866 | 26.21938 | 28.30226 |
| | 40 | 18.55040 | 21.02736 | 23.33819 | 26.21881 | 28.30161 |
| | 45 | 18.55018 | 21.02708 | 23.33786 | 26.21842 | 28.30116 |
| | 50 | 18.55002 | 21.02689 | 23.33763 | 26.21814 | 28.30085 |
| | 60 | 18.54981 | 21.02664 | 23.33733 | 26.21778 | 28.30044 |
| | 70 | 18.54969 | 21.02648 | 23.33715 | 26.21756 | 28.30019 |
| | 80 | 18.54961 | 21.02639 | 23.33704 | 26.21742 | 28.30003 |
| | 90 | 18.54955 | 21.02632 | 23.33696 | 26.21732 | 28.29992 |
| | 100 | 18.54951 | 21.02627 | 23.33690 | 26.21726 | 28.29985 |
| $L_o$ | 10 | .816032 | .794165 | .774291 | .750208 | .733261 |
| | 11 | .832598 | .812468 | .794126 | .771836 | .756110 |
| | 12 | .846432 | .827789 | .810766 | .790033 | .775372 |
| | 13 | .858157 | .840801 | .824925 | .805552 | .791827 |
| | 14 | .868221 | .851988 | .837117 | .818941 | .806044 |
| | 15 | .876953 | .861709 | .847725 | .830610 | .818450 |
| | 16 | .884601 | .870233 | .857038 | .840869 | .829368 |
| | 17 | .891354 | .877768 | .865279 | .849958 | .839050 |
| | 18 | .897362 | .884477 | .872623 | .858068 | .847695 |
| | 19 | .902740 | .890488 | .879208 | .865346 | .855460 |
| | 20 | .907583 | .895906 | .885147 | .871916 | .862473 |
| | 25 | .926007 | .916550 | .907814 | .897041 | .889331 |
| | 30 | .938308 | .930364 | .923013 | .913930 | .907419 |
| | 35 | .947103 | .940254 | .933910 | .926062 | .920428 |
| | 40 | .953703 | .947686 | .942106 | .935197 | .930232 |
| | 45 | .958839 | .953473 | .948494 | .942323 | .937886 |
| | 50 | .962949 | .958107 | .953612 | .948038 | .944027 |
| | 60 | .969118 | .965067 | .961303 | .956632 | .9532[illegible] |
| | 70 | .973525 | .970044 | .966807 | .962787 | .959891 |
| | 80 | .976832 | .973779 | .970940 | .967412 | .964870 |
| | 90 | .979404 | .976686 | .974158 | .971015 | .968749 |
| | 100 | .981462 | .979013 | .976734 | .973900 | .971856 |

Table A5 Percentage Points of $L_o$ and $M_o = -2s \log \lambda_o$

P = 6

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 22.33283 | 25.02648 | 27.52407 | 30.62023 | 32.84873 |
| | 11 | 22.32808 | 25.02082 | 27.51749 | 30.61244 | 32.84002 |
| | 12 | 22.32454 | 25.01659 | 27.51258 | 30.60662 | 32.83350 |
| | 13 | 22.32183 | 25.01335 | 27.50882 | 30.60216 | 32.82850 |
| | 14 | 22.31970 | 25.01081 | 27.50587 | 30.59866 | 32.82458 |
| | 15 | 22.31800 | 25.00879 | 27.50351 | 30.59587 | 32.82145 |
| | 16 | 22.31663 | 25.00715 | 27.50161 | 30.59361 | 32.81892 |
| | 17 | 22.31550 | 25.00580 | 27.50004 | 30.59174 | 32.81683 |
| | 18 | 22.31456 | 25.00468 | 27.49873 | 30.59020 | 32.81509 |
| | 19 | 22.31377 | 25.00373 | 27.49764 | 30.58889 | 32.81363 |
| | 20 | 22.31310 | 25.00293 | 27.49670 | 30.58779 | 32.81240 |
| | 25 | 22.31090 | 25.00030 | 27.49364 | 30.58415 | 32.80832 |
| | 30 | 22.30972 | 24.99889 | 27.49200 | 30.58221 | 32.80614 |
| | 35 | 22.30902 | 24.99805 | 27.49103 | 30.58105 | 32.80484 |
| | 40 | 22.30857 | 24.99751 | 27.49040 | 30.58030 | 32.80400 |
| | 45 | 22.30826 | 24.99715 | 27.48997 | 30.57979 | 32.80343 |
| | 50 | 22.30805 | 24.99689 | 27.48967 | 30.57943 | 32.80302 |
| | 60 | 22.30776 | 24.99655 | 27.48928 | 30.57896 | 32.80250 |
| | 70 | 22.30759 | 24.99635 | 27.48904 | 30.57868 | 32.80218 |
| | 80 | 22.30748 | 24.99621 | 27.48889 | 30.57850 | 32.80198 |
| | 90 | 22.30741 | 24.99612 | 27.48878 | 30.57838 | 32.80184 |
| | 100 | 22.30736 | 24.99606 | 27.48871 | 30.57829 | 32.80174 |
| $L_o$ | 10 | .816090 | .796329 | .778433 | .756807 | .741614 |
| | 11 | .832615 | .814423 | .797911 | .777905 | .763818 |
| | 12 | .846420 | .829572 | .814250 | .795649 | .782525 |
| | 13 | .858125 | .842438 | .828151 | .810776 | .798497 |
| | 14 | .868174 | .853502 | .840120 | .823824 | .812291 |
| | 15 | .876895 | .863116 | .850534 | .835192 | .824322 |
| | 16 | .884535 | .871547 | .859676 | .845185 | .834908 |
| | 17 | .891282 | .879001 | .867765 | .854038 | .844293 |
| | 18 | .897285 | .885638 | .874974 | .861934 | .852670 |
| | 19 | .902660 | .891585 | .881438 | .869021 | .860193 |
| | 20 | .907501 | .896945 | .887268 | .875417 | .866987 |
| | 25 | .925924 | .917373 | .909517 | .899872 | .892994 |
| | 30 | .938228 | .931045 | .924434 | .916306 | .910500 |
| | 35 | .947028 | .940835 | .935130 | .928107 | .923085 |
| | 40 | .953633 | .948192 | .943174 | .936993 | .932569 |
| | 45 | .958774 | .953921 | .949444 | .943924 | .939971 |
| | 50 | .962889 | .958510 | .954468 | .949481 | .945909 |
| | 60 | .969064 | .965401 | .962017 | .957839 | .954843 |
| | 70 | .973478 | .970329 | .967419 | .963824 | .961245 |
| | 80 | .976789 | .974028 | .971476 | .968321 | .966057 |
| | 90 | .979366 | .976908 | .974634 | .971824 | .969806 |
| | 100 | .981427 | .979212 | .977163 | .974629 | .972809 |

Table A6 Percentage Points of $L_o$ and $M_o = -2s \log \lambda_o$

P = 7

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 26.02186 | 28.90750 | 31.57028 | 34.85673 | 37.21360 |
| | 11 | 26.01587 | 28.90046 | 31.56220 | 34.84727 | 37.20310 |
| | 12 | 26.01140 | 28.89520 | 31.55615 | 34.84020 | 37.19524 |
| | 13 | 26.00798 | 28.89117 | 31.55152 | 34.83478 | 37.18922 |
| | 14 | 26.00530 | 28.88801 | 31.54789 | 34.83052 | 37.18449 |
| | 15 | 26.00316 | 28.88549 | 31.54500 | 34.82713 | 37.18072 |
| | 16 | 26.00142 | 28.88345 | 31.54265 | 34.82438 | 37.17766 |
| | 17 | 26.00000 | 28.88177 | 31.54072 | 34.82212 | 37.17515 |
| | 18 | 25.99881 | 28.88037 | 31.53911 | 34.82024 | 37.17306 |
| | 19 | 25.99781 | 28.87919 | 31.53776 | 34.81865 | 37.17130 |
| | 20 | 25.99697 | 28.87820 | 31.53661 | 34.81731 | 37.16980 |
| | 25 | 25.99418 | 28.87492 | 31.53284 | 34.81289 | 37.16488 |
| | 30 | 25.99270 | 28.87316 | 31.53082 | 34.81052 | 37.16226 |
| | 35 | 25.99181 | 28.87212 | 31.52962 | 34.80911 | 37.16069 |
| | 40 | 25.99124 | 28.87145 | 31.52885 | 34.80821 | 37.15968 |
| | 45 | 25.99086 | 28.87099 | 31.52833 | 34.80759 | 37.15899 |
| | 50 | 25.99058 | 28.87067 | 31.52795 | 34.80715 | 37.15851 |
| | 60 | 25.99022 | 28.87024 | 31.52747 | 34.80658 | 37.15787 |
| | 70 | 25.99001 | 28.86999 | 31.52717 | 34.80624 | 37.15749 |
| | 80 | 25.98987 | 28.86983 | 31.52699 | 34.80602 | 37.15725 |
| | 90 | 25.98978 | 28.86972 | 31.52686 | 34.80587 | 37.15708 |
| | 100 | 25.98971 | 28.86964 | 31.52677 | 34.80576 | 37.15696 |
| $L_o$ | 10 | .816611 | .798470 | .782087 | .762330 | .748469 |
| | 11 | .833069 | .816369 | .801255 | .782988 | .770145 |
| | 12 | .846821 | .831354 | .817333 | .800355 | .788397 |
| | 13 | .858482 | .844083 | .831010 | .815156 | .803973 |
| | 14 | .868496 | .855027 | .842786 | .827919 | .817420 |
| | 15 | .877187 | .864538 | .853029 | .839037 | .829145 |
| | 16 | .884802 | .872880 | .862022 | .848808 | .839458 |
| | 17 | .891528 | .880254 | .869979 | .857463 | .848599 |
| | 18 | .897513 | .886821 | .877069 | .865181 | .856757 |
| | 19 | .902872 | .892705 | .883426 | .872108 | .864082 |
| | 20 | .907699 | .898008 | .889159 | .878359 | .870695 |
| | 25 | .926071 | .918222 | .911039 | .902252 | .896004 |
| | 30 | .938344 | .931750 | .925707 | .918304 | .913032 |
| | 35 | .947124 | .941439 | .936224 | .929829 | .925270 |
| | 40 | .953715 | .948719 | .944133 | .938505 | .934490 |
| | 45 | .958845 | .954390 | .950298 | .945272 | .941685 |
| | 50 | .962951 | .958931 | .955237 | .950698 | .947456 |
| | 60 | .969115 | .965752 | .962659 | .958856 | .956138 |
| | 70 | .973520 | .970629 | .967970 | .964698 | .962359 |
| | 80 | .976826 | .974291 | .971958 | .969087 | .967034 |
| | 90 | .979398 | .977141 | .975063 | .972506 | .970676 |
| | 100 | .981456 | .979422 | .977549 | .975243 | .973593 |

Table A7 Percentage Points of $L_o$ and $M_o = -2s \log \lambda_o$

P = 8

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 29.65415 | 32.71608 | 35.53070 | 38.99227 | 41.46741 |
| | 11 | 29.64695 | 32.70770 | 35.52116 | 38.98121 | 41.45522 |
| | 12 | 29.64157 | 32.70143 | 35.51403 | 38.97295 | 41.44610 |
| | 13 | 29.63745 | 32.69663 | 35.50856 | 38.96661 | 41.43910 |
| | 14 | 29.63422 | 32.69287 | 35.50428 | 38.96165 | 41.43362 |
| | 15 | 29.63164 | 32.68987 | 35.50086 | 38.95768 | 41.42925 |
| | 16 | 29.62955 | 32.68743 | 35.49809 | 38.95447 | 41.42570 |
| | 17 | 29.62783 | 32.68543 | 35.49581 | 38.95182 | 41.42278 |
| | 18 | 29.62640 | 32.68377 | 35.49391 | 38.94962 | 41.42035 |
| | 19 | 29.62520 | 32.68237 | 35.49232 | 38.94777 | 41.41830 |
| | 20 | 29.62418 | 32.68118 | 35.49096 | 38.94620 | 41.41657 |
| | 25 | 29.62083 | 32.67727 | 35.48651 | 38.94103 | 41.41086 |
| | 30 | 29.61904 | 32.67518 | 35.48413 | 38.93827 | 41.40781 |
| | 35 | 29.61797 | 32.67394 | 35.48271 | 38.93663 | 41.40599 |
| | 40 | 29.61729 | 32.67314 | 35.48180 | 38.93557 | 41.40482 |
| | 45 | 29.61682 | 32.67259 | 35.48118 | 38.93484 | 41.40402 |
| | 50 | 29.61649 | 32.67220 | 35.48073 | 38.93433 | 41.40345 |
| | 60 | 29.61605 | 32.67170 | 35.48016 | 38.93366 | 41.40271 |
| | 70 | 29.61580 | 32.67140 | 35.47982 | 38.93327 | 41.40227 |
| | 80 | 29.61563 | 32.67120 | 35.47959 | 38.93301 | 41.40199 |
| | 90 | 29.61552 | 32.67107 | 35.47944 | 38.93283 | 41.40179 |
| | 100 | 29.61543 | 32.67098 | 35.47933 | 38.93271 | 41.40165 |
| $L_o$ | 10 | .817317 | .800468 | .785288 | .767012 | .754205 |
| | 11 | .833699 | .818190 | .804189 | .787298 | .775439 |
| | 12 | .847389 | .833027 | .820040 | .804346 | .793309 |
| | 13 | .859000 | .845629 | .833522 | .818871 | .808554 |
| | 14 | .868970 | .856464 | .845128 | .831393 | .821710 |
| | 15 | .877625 | .865881 | .855224 | .842299 | .833178 |
| | 16 | .885208 | .874139 | .864086 | .851883 | .843263 |
| | 17 | .891907 | .881440 | .871927 | .860370 | .852201 |
| | 18 | .897868 | .887941 | .878914 | .867938 | .860175 |
| | 19 | .903206 | .893767 | .885178 | .874729 | .867334 |
| | 20 | .908014 | .899017 | .890826 | .880857 | .873797 |
| | 25 | .926316 | .919029 | .912382 | .904274 | .898521 |
| | 30 | .938545 | .932423 | .926832 | .920002 | .915150 |
| | 35 | .947294 | .942016 | .937191 | .931292 | .927098 |
| | 40 | .953862 | .949224 | .944982 | .939791 | .936097 |
| | 45 | .958975 | .954839 | .951053 | .946418 | .943119 |
| | 50 | .963067 | .959335 | .955918 | .951732 | .948751 |
| | 60 | .969210 | .966088 | .963227 | .959721 | .957222 |
| | 70 | .973602 | .970918 | .968458 | .965441 | .963290 |
| | 80 | .976897 | .974543 | .972386 | .969739 | .967851 |
| | 90 | .979460 | .977365 | .975444 | .973086 | .971404 |
| | 100 | .981512 | .979624 | .977892 | .975766 | .974249 |

Table A8 Percentage Points of $L_o$ and $M_o = -2s \log \lambda_o$

P = 9

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 33.24183 | 36.46767 | 39.42357 | 43.04824 | 45.63364 |
| | 11 | 33.23343 | 36.45797 | 39.41262 | 43.03566 | 45.61983 |
| | 12 | 33.22715 | 36.45073 | 39.40444 | 43.02625 | 45.60951 |
| | 13 | 33.22234 | 36.44518 | 39.39816 | 43.01904 | 45.60159 |
| | 14 | 33.21857 | 36.44082 | 39.39325 | 43.01338 | 45.59538 |
| | 15 | 33.21557 | 36.43735 | 39.38932 | 43.00887 | 45.59043 |
| | 16 | 33.21313 | 36.43453 | 39.38614 | 43.00521 | 45.58641 |
| | 17 | 33.21112 | 36.43222 | 39.38352 | 43.00220 | 45.58310 |
| | 18 | 33.20946 | 36.43029 | 39.38134 | 42.99969 | 45.58035 |
| | 19 | 33.20805 | 36.42867 | 39.37951 | 42.99759 | 45.57804 |
| | 20 | 33.20686 | 36.42730 | 39.37796 | 42.99580 | 45.57607 |
| | 25 | 33.20295 | 36.42278 | 39.37284 | 42.98991 | 45.56960 |
| | 30 | 33.20086 | 36.42036 | 39.37011 | 42.98677 | 45.56615 |
| | 35 | 33.19961 | 36.41892 | 39.36848 | 42.98489 | 45.56409 |
| | 40 | 33.19881 | 36.41799 | 39.36743 | 42.98369 | 45.56276 |
| | 45 | 33.19826 | 36.41736 | 39.36672 | 42.98286 | 45.56186 |
| | 50 | 33.19788 | 36.41691 | 39.36621 | 42.98228 | 45.56121 |
| | 60 | 33.19737 | 36.41633 | 39.36555 | 42.98152 | 45.56038 |
| | 70 | 33.19707 | 36.41598 | 39.36516 | 42.98107 | 45.55988 |
| | 80 | 33.19688 | 36.41576 | 39.36490 | 42.98077 | 45.55956 |
| | 90 | 33.19674 | 36.41560 | 39.36473 | 42.98057 | 45.55934 |
| | 100 | 33.19665 | 36.41549 | 39.36460 | 42.98043 | 45.55918 |
| $L_o$ | 10 | .818085 | .802299 | .788101 | .771034 | .759087 |
| | 11 | .834391 | .819861 | .806769 | .791001 | .779943 |
| | 12 | .848018 | .834563 | .822422 | .807776 | .797489 |
| | 13 | .859576 | .847050 | .835734 | .822064 | .812451 |
| | 14 | .869502 | .857787 | .847192 | .834380 | .825360 |
| | 15 | .878118 | .867117 | .857159 | .845104 | .836610 |
| | 16 | .885668 | .875300 | .865906 | .854526 | .846501 |
| | 17 | .892338 | .882534 | .873645 | .862869 | .855265 |
| | 18 | .898273 | .888975 | .880541 | .870308 | .863083 |
| | 19 | .903588 | .894748 | .886723 | .876983 | .870101 |
| | 20 | .908376 | .899950 | .892298 | .883004 | .876436 |
| | 25 | .926602 | .919777 | .913569 | .906013 | .900663 |
| | 30 | .938781 | .933048 | .927826 | .921463 | .916951 |
| | 35 | .947494 | .942552 | .938046 | .932551 | .928652 |
| | 40 | .954036 | .949693 | .945732 | .940897 | .937464 |
| | 45 | .959129 | .955256 | .951721 | .947405 | .944339 |
| | 50 | .963206 | .959711 | .956520 | .952622 | .949852 |
| | 60 | .969325 | .966402 | .963730 | .960466 | .958144 |
| | 70 | .973700 | .971187 | .968890 | .966081 | .964083 |
| | 80 | .976982 | .974779 | .972764 | .970300 | .968546 |
| | 90 | .979536 | .977574 | .975780 | .973585 | .972023 |
| | 100 | .981580 | .979812 | .978195 | .976216 | .974807 |

Table A9 Percentage Points of $L_o$ and $M_o = -2s \log \lambda_o$

P = 10

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_o$ | 10 | 36.79324 | 40.17289 | 43.26148 | 47.03941 | 49.72850 |
| | 11 | 36.78366 | 40.16192 | 43.24915 | 47.02535 | 49.71314 |
| | 12 | 36.77650 | 40.15371 | 43.23995 | 47.01484 | 49.70166 |
| | 13 | 36.77101 | 40.14742 | 43.23288 | 47.00678 | 49.69285 |
| | 14 | 36.76671 | 40.14250 | 43.22735 | 47.00046 | 49.68594 |
| | 15 | 36.76327 | 40.13856 | 43.22293 | 46.99542 | 49.68043 |
| | 16 | 36.76049 | 40.13537 | 43.21935 | 46.99132 | 49.67595 |
| | 17 | 36.75820 | 40.13275 | 43.21640 | 46.98796 | 49.67228 |
| | 18 | 36.75630 | 40.13057 | 43.21395 | 46.98516 | 49.66921 |
| | 19 | 36.75470 | 40.12873 | 43.21189 | 46.98281 | 49.66664 |
| | 20 | 36.75334 | 40.12718 | 43.21014 | 46.98081 | 49.66445 |
| | 25 | 36.74887 | 40.12205 | 43.20438 | 46.97423 | 49.65726 |
| | 30 | 36.74648 | 40.11931 | 43.20130 | 46.97071 | 49.65341 |
| | 35 | 36.74506 | 40.11768 | 43.19947 | 46.96861 | 49.65112 |
| | 40 | 36.74415 | 40.11663 | 43.19829 | 46.96726 | 49.64964 |
| | 45 | 36.74352 | 40.11592 | 43.19749 | 46.96635 | 49.64864 |
| | 50 | 36.74308 | 40.11541 | 43.19691 | 46.96569 | 49.64792 |
| | 60 | 36.74250 | 40.11475 | 43.19617 | 46.96484 | 49.64699 |
| | 70 | 36.74216 | 40.11435 | 43.19573 | 46.96433 | 49.64644 |
| | 80 | 36.74194 | 40.11410 | 43.19544 | 46.96401 | 49.64608 |
| | 90 | 36.74179 | 40.11392 | 43.19524 | 46.96378 | 49.64583 |
| | 100 | 36.74168 | 40.11380 | 43.19510 | 46.96362 | 49.64566 |
| $L_o$ | 10 | .818860 | .803965 | .790591 | .774533 | .763303 |
| | 11 | .835092 | .821384 | .809054 | .794223 | .783833 |
| | 12 | .848658 | .835965 | .824532 | .810760 | .801098 |
| | 13 | .860164 | .848349 | .837693 | .824842 | .815816 |
| | 14 | .870046 | .858996 | .849021 | .836978 | .828511 |
| | 15 | .878625 | .868248 | .858873 | .847544 | .839572 |
| | 16 | .886142 | .876362 | .867520 | .856826 | .849296 |
| | 17 | .892782 | .883535 | .875169 | .865044 | .857909 |
| | 18 | .898692 | .889922 | .881984 | .872371 | .865593 |
| | 19 | .903984 | .895646 | .888094 | .878944 | .872489 |
| | 20 | .908751 | .900804 | .893603 | .884874 | .878713 |
| | 25 | .926900 | .920464 | .914622 | .907527 | .902511 |
| | 30 | .939028 | .933621 | .928708 | .922734 | .918506 |
| | 35 | .947705 | .943044 | .938806 | .933647 | .929994 |
| | 40 | .954220 | .950125 | .946398 | .941860 | .938643 |
| | 45 | .959292 | .955640 | .952315 | .948264 | .945391 |
| | 50 | .963352 | .960057 | .957055 | .953397 | .950802 |
| | 60 | .969447 | .966690 | .964178 | .961114 | .958939 |
| | 70 | .973804 | .971434 | .969274 | .966638 | .964767 |
| | 80 | .977073 | .974995 | .973101 | .970788 | .969146 |
| | 90 | .979617 | .977767 | .976080 | .974020 | .972557 |
| | 100 | .981653 | .979986 | .978465 | .976608 | .975289 |

Table A10 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 2

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 2.70345 | 3.83790 | 5.01843 | 6.62625 | 7.86785 |
| | 11 | 2.70386 | 3.83859 | 5.01949 | 6.62792 | 7.87009 |
| | 12 | 2.70416 | 3.83910 | 5.02026 | 6.62916 | 7.87174 |
| | 13 | 2.70438 | 3.83948 | 5.02085 | 6.63009 | 7.87299 |
| | 14 | 2.70456 | 3.83978 | 5.02131 | 6.63081 | 7.87396 |
| | 15 | 2.70469 | 3.84001 | 5.02167 | 6.63138 | 7.87473 |
| | 16 | 2.70481 | 3.84020 | 5.02196 | 6.63184 | 7.87534 |
| | 17 | 2.70490 | 3.84036 | 5.02220 | 6.63222 | 7.87585 |
| | 18 | 2.70497 | 3.84048 | 5.02239 | 6.63253 | 7.87626 |
| | 19 | 2.70503 | 3.84059 | 5.02256 | 6.63279 | 7.87661 |
| | 20 | 2.70509 | 3.84068 | 5.02269 | 6.63301 | 7.87690 |
| | 25 | 2.70526 | 3.84097 | 5.02314 | 6.63372 | 7.87786 |
| | 30 | 2.70535 | 3.84113 | 5.02338 | 6.63409 | 7.87836 |
| | 35 | 2.70540 | 3.84122 | 5.02352 | 6.63431 | 7.87866 |
| | 40 | 2.70544 | 3.84128 | 5.02361 | 6.63445 | 7.87884 |
| | 45 | 2.70546 | 3.84132 | 5.02367 | 6.63455 | 7.87897 |
| | 50 | 2.70548 | 3.84134 | 5.02371 | 6.63462 | 7.87906 |
| | 60 | 2.70550 | 3.84138 | 5.02377 | 6.63470 | 7.87918 |
| | 70 | 2.70551 | 3.84140 | 5.02380 | 6.63476 | 7.87925 |
| | 80 | 2.70552 | 3.84142 | 5.02382 | 6.63479 | 7.87930 |
| | 90 | 2.70552 | 3.84142 | 5.02383 | 6.63481 | 7.87933 |
| | 100 | 2.70553 | 3.84143 | 5.02384 | 6.63483 | 7.87935 |
| $L_1$ | 10 | .925666 | .896144 | .866422 | .827521 | .798679 |
| | 11 | .933019 | .906263 | .879234 | .843710 | .817261 |
| | 12 | .939049 | .914588 | .889807 | .857129 | .832715 |
| | 13 | .944084 | .921557 | .898682 | .868432 | .845768 |
| | 14 | .948351 | .927475 | .906235 | .878082 | .856937 |
| | 15 | .952013 | .932563 | .912741 | .886415 | .866601 |
| | 16 | .955191 | .936985 | .918404 | .893683 | .875045 |
| | 17 | .957974 | .940863 | .923377 | .900079 | .882485 |
| | 18 | .960431 | .944291 | .927780 | .905749 | .889091 |
| | 19 | .962618 | .947344 | .931704 | .910811 | .894994 |
| | 20 | .964575 | .950080 | .935224 | .915358 | .900302 |
| | 25 | .971925 | .960375 | .948498 | .932554 | .920420 |
| | 30 | .976750 | .967151 | .957259 | .943945 | .933787 |
| | 35 | .980159 | .971948 | .963473 | .952045 | .943310 |
| | 40 | .982697 | .975522 | .968109 | .958100 | .950439 |
| | 45 | .984659 | .978289 | .971702 | .962798 | .955976 |
| | 50 | .986222 | .980494 | .974566 | .966549 | .960400 |
| | 60 | .988553 | .983787 | .978849 | .972162 | .967027 |
| | 70 | .990210 | .986128 | .981898 | .976162 | .971755 |
| | 80 | .991448 | .987879 | .984178 | .979157 | .975297 |
| | 90 | .992408 | .989237 | .985948 | .981484 | .978049 |
| | 100 | .993174 | .990322 | .987362 | .983343 | .980250 |

Table A11 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 3

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 4.60199 | 5.98666 | 7.37102 | 9.20057 | 10.58421 |
| | 11 | 4.60261 | 5.98759 | 7.37232 | 9.20246 | 10.58662 |
| | 12 | 4.60306 | 5.98827 | 7.37329 | 9.20385 | 10.58839 |
| | 13 | 4.60340 | 5.98879 | 7.37401 | 9.20491 | 10.58973 |
| | 14 | 4.60367 | 5.98920 | 7.37458 | 9.20573 | 10.59077 |
| | 15 | 4.60388 | 5.98951 | 7.37502 | 9.20637 | 10.59159 |
| | 16 | 4.60405 | 5.98977 | 7.37538 | 9.20689 | 10.59225 |
| | 17 | 4.60419 | 5.98998 | 7.37567 | 9.20731 | 10.59279 |
| | 18 | 4.60430 | 5.99015 | 7.37591 | 9.20766 | 10.59323 |
| | 19 | 4.60440 | 5.99029 | 7.37612 | 9.20796 | 10.59360 |
| | 20 | 4.60448 | 5.99041 | 7.37629 | 9.20820 | 10.59392 |
| | 25 | 4.60474 | 5.99081 | 7.37684 | 9.20901 | 10.59494 |
| | 30 | 4.60487 | 5.99102 | 7.37713 | 9.20943 | 10.59548 |
| | 35 | 4.60496 | 5.99114 | 7.37730 | 9.20968 | 10.59580 |
| | 40 | 4.60501 | 5.99122 | 7.37741 | 9.20984 | 10.59600 |
| | 45 | 4.60504 | 5.99127 | 7.37749 | 9.20995 | 10.59614 |
| | 50 | 4.60507 | 5.99131 | 7.37754 | 9.21002 | 10.59623 |
| | 60 | 4.60510 | 5.99136 | 7.37761 | 9.21012 | 10.59636 |
| | 70 | 4.60512 | 5.99139 | 7.37765 | 9.21018 | 10.59643 |
| | 80 | 4.60513 | 5.99141 | 7.37768 | 9.21022 | 10.59648 |
| | 90 | 4.60514 | 5.99142 | 7.37769 | 9.21025 | 10.59651 |
| | 100 | 4.60515 | 5.99143 | 7.37771 | 9.21026 | 10.59654 |
| $L_1$ | 10 | .916329 | .892552 | .869397 | .839714 | .817940 |
| | 11 | .924545 | .902974 | .881911 | .854824 | .834891 |
| | 12 | .931293 | .911556 | .892240 | .867337 | .848965 |
| | 13 | .936934 | .918745 | .900911 | .877869 | .860833 |
| | 14 | .941720 | .924854 | .908291 | .886854 | .870976 |
| | 15 | .945830 | .930109 | .914650 | .894610 | .879744 |
| | 16 | .949400 | .934678 | .920185 | .901372 | .887398 |
| | 17 | .952528 | .938686 | .925046 | .907320 | .894138 |
| | 18 | .955292 | .942231 | .929349 | .912592 | .900117 |
| | 19 | .957752 | .945389 | .933186 | .917296 | .905457 |
| | 20 | .959956 | .948219 | .936627 | .921521 | .910257 |
| | 25 | .968239 | .958878 | .949607 | .937490 | .928427 |
| | 30 | .973683 | .965898 | .958175 | .948061 | .940481 |
| | 35 | .977534 | .970871 | .964253 | .955575 | .949061 |
| | 40 | .980402 | .974579 | .968789 | .961189 | .955480 |
| | 45 | .982621 | .977449 | .972304 | .965544 | .960462 |
| | 50 | .984388 | .979737 | .975107 | .969021 | .964442 |
| | 60 | .987027 | .983155 | .979298 | .974222 | .970401 |
| | 70 | .988903 | .985586 | .982281 | .977929 | .974649 |
| | 80 | .990304 | .987404 | .984513 | .980703 | .977831 |
| | 90 | .991392 | .988815 | .986245 | .982858 | .980303 |
| | 100 | .992260 | .989942 | .987629 | .984580 | .982280 |

Table A12 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 4

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 6.24741 | 7.80906 | 9.34081 | 11.33436 | 12.82518 |
| | 11 | 6.24818 | 7.81016 | 9.34228 | 11.33640 | 12.82770 |
| | 12 | 6.24875 | 7.81097 | 9.34337 | 11.33790 | 12.82955 |
| | 13 | 6.24918 | 7.81158 | 9.34419 | 11.33903 | 12.83095 |
| | 14 | 6.24951 | 7.81206 | 9.34482 | 11.33991 | 12.83203 |
| | 15 | 6.24978 | 7.81243 | 9.34532 | 11.34061 | 12.83289 |
| | 16 | 6.24999 | 7.81273 | 9.34573 | 11.34116 | 12.83358 |
| | 17 | 6.25016 | 7.81298 | 9.34606 | 11.34162 | 12.83414 |
| | 18 | 6.25030 | 7.81318 | 9.34633 | 11.34200 | 12.83461 |
| | 19 | 6.25042 | 7.81335 | 9.34656 | 11.34231 | 12.83500 |
| | 20 | 6.25052 | 7.81349 | 9.34675 | 11.34258 | 12.83533 |
| | 25 | 6.25085 | 7.81396 | 9.34737 | 11.34344 | 12.83639 |
| | 30 | 6.25102 | 7.81420 | 9.34770 | 11.34389 | 12.83695 |
| | 35 | 6.25112 | 7.81435 | 9.34789 | 11.34416 | 12.83728 |
| | 40 | 6.25119 | 7.81444 | 9.34802 | 11.34433 | 12.83749 |
| | 45 | 6.25123 | 7.81450 | 9.34810 | 11.34445 | 12.83764 |
| | 50 | 6.25126 | 7.81455 | 9.34816 | 11.34453 | 12.83774 |
| | 60 | 6.25130 | 7.81460 | 9.34824 | 11.34463 | 12.83787 |
| | 70 | 6.25132 | 7.81464 | 9.34828 | 11.34470 | 12.83795 |
| | 80 | 6.25134 | 7.81466 | 9.34831 | 11.34474 | 12.83800 |
| | 90 | 6.25135 | 7.81467 | 9.34833 | 11.34477 | 12.83803 |
| | 100 | 6.25136 | 7.81468 | 9.34834 | 11.34478 | 12.83805 |
| $L_1$ | 10 | .915005 | .894913 | .875633 | .851163 | .833311 |
| | 11 | .923334 | .905105 | .887574 | .865265 | .848948 |
| | 12 | .930178 | .913498 | .897426 | .876931 | .861910 |
| | 13 | .935901 | .920528 | .905693 | .886742 | .872828 |
| | 14 | .940757 | .926502 | .912728 | .895106 | .882149 |
| | 15 | .944930 | .931641 | .918787 | .902321 | .890199 |
| | 16 | .948554 | .936109 | .924060 | .908609 | .897221 |
| | 17 | .951730 | .940029 | .928691 | .914137 | .903400 |
| | 18 | .954538 | .943497 | .932790 | .919034 | .908879 |
| | 19 | .957036 | .946585 | .936443 | .923404 | .913771 |
| | 20 | .959275 | .949353 | .939720 | .927327 | .918164 |
| | 25 | .967692 | .959777 | .952076 | .942143 | .934782 |
| | 30 | .973226 | .966643 | .960229 | .951943 | .945793 |
| | 35 | .977142 | .971508 | .966012 | .958905 | .953623 |
| | 40 | .980058 | .975134 | .970327 | .964105 | .959477 |
| | 45 | .982315 | .977941 | .973669 | .968137 | .964019 |
| | 50 | .984112 | .980179 | .976335 | .971355 | .967646 |
| | 60 | .986797 | .983522 | .980321 | .976168 | .973074 |
| | 70 | .988705 | .985901 | .983157 | .979597 | .976942 |
| | 80 | .990132 | .987679 | .985279 | .982163 | .979839 |
| | 90 | .991238 | .989059 | .986926 | .984156 | .982089 |
| | 100 | .992121 | .990161 | .988241 | .985748 | .983887 |

Table A13 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 5

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 7.77478 | 9.48133 | 11.13497 | 13.26554 | 14.84673 |
| | 11 | 7.77568 | 9.48258 | 11.13658 | 13.26771 | 14.84936 |
| | 12 | 7.77635 | 9.48349 | 11.13777 | 13.26931 | 14.85129 |
| | 13 | 7.77685 | 9.48418 | 11.13867 | 13.27051 | 14.85275 |
| | 14 | 7.77724 | 9.48471 | 11.13936 | 13.27144 | 14.85388 |
| | 15 | 7.77755 | 9.48514 | 11.13991 | 13.27218 | 14.85478 |
| | 16 | 7.77780 | 9.48548 | 11.14036 | 13.27277 | 14.85550 |
| | 17 | 7.77800 | 9.48575 | 11.14072 | 13.27326 | 14.85608 |
| | 18 | 7.77817 | 9.48598 | 11.14101 | 13.27366 | 14.85657 |
| | 19 | 7.77831 | 9.48617 | 11.14126 | 13.27399 | 14.85697 |
| | 20 | 7.77842 | 9.48633 | 11.14147 | 13.27427 | 14.85731 |
| | 25 | 7.77881 | 9.48686 | 11.14216 | 13.27519 | 14.85842 |
| | 30 | 7.77901 | 9.48714 | 11.14252 | 13.27567 | 14.85901 |
| | 35 | 7.77913 | 9.48730 | 11.14273 | 13.27595 | 14.85935 |
| | 40 | 7.77920 | 9.48740 | 11.14286 | 13.27614 | 14.85957 |
| | 45 | 7.77925 | 9.48747 | 11.14295 | 13.27626 | 14.85972 |
| | 50 | 7.77929 | 9.48752 | 11.14302 | 13.27635 | 14.85982 |
| | 60 | 7.77934 | 9.48759 | 11.14310 | 13.27646 | 14.85996 |
| | 70 | 7.77937 | 9.48763 | 11.14315 | 13.27652 | 14.86004 |
| | 80 | 7.77938 | 9.48765 | 11.14318 | 13.27657 | 14.86009 |
| | 90 | 7.77940 | 9.48767 | 11.14321 | 13.27660 | 14.86013 |
| | 100 | 7.77940 | 9.48768 | 11.14322 | 13.27662 | 14.86015 |
| $L_1$ | 10 | .915441 | .897859 | .881145 | .860067 | .844751 |
| | 11 | .923722 | .907773 | .892581 | .873380 | .859397 |
| | 12 | .930528 | .915935 | .902012 | .884384 | .871524 |
| | 13 | .936219 | .922771 | .909923 | .893631 | .881729 |
| | 14 | .941049 | .928580 | .916653 | .901511 | .890434 |
| | 15 | .945200 | .933576 | .922448 | .908305 | .897948 |
| | 16 | .948804 | .937920 | .927491 | .914223 | .904498 |
| | 17 | .951964 | .941730 | .931918 | .919424 | .910259 |
| | 18 | .954757 | .945100 | .935836 | .924031 | .915365 |
| | 19 | .957243 | .948102 | .939327 | .928140 | .919922 |
| | 20 | .959470 | .950793 | .942459 | .931827 | .924014 |
| | 25 | .967844 | .960923 | .954263 | .945749 | .939479 |
| | 30 | .973351 | .967595 | .962050 | .954950 | .949715 |
| | 35 | .977248 | .972321 | .967571 | .961483 | .956990 |
| | 40 | .980150 | .975845 | .971690 | .966362 | .962426 |
| | 45 | .982396 | .978572 | .974880 | .970144 | .966643 |
| | 50 | .984185 | .980746 | .977425 | .973161 | .970009 |
| | 60 | .986857 | .983994 | .981228 | .977674 | .975045 |
| | 70 | .988757 | .986305 | .983934 | .980888 | .978633 |
| | 80 | .990176 | .988032 | .985958 | .983293 | .981319 |
| | 90 | .991278 | .989373 | .987530 | .985160 | .983405 |
| | 100 | .992157 | .990443 | .988785 | .986652 | .985072 |

Table A14 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 6

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 9.23107 | 11.06343 | 12.82351 | 15.07448 | 16.73551 |
| | 11 | 9.23210 | 11.06481 | 12.82526 | 15.07677 | 16.73825 |
| | 12 | 9.23285 | 11.06582 | 12.82654 | 15.07846 | 16.74026 |
| | 13 | 9.23342 | 11.06658 | 12.82751 | 15.07973 | 16.74179 |
| | 14 | 9.23386 | 11.06717 | 12.82826 | 15.08072 | 16.74296 |
| | 15 | 9.23421 | 11.06764 | 12.82886 | 15.08150 | 16.74390 |
| | 16 | 9.23449 | 11.06801 | 12.82934 | 15.08212 | 16.74464 |
| | 17 | 9.23472 | 11.06832 | 12.82972 | 15.08263 | 16.74525 |
| | 18 | 9.23491 | 11.06857 | 12.83005 | 15.08305 | 16.74576 |
| | 19 | 9.23507 | 11.06878 | 12.83032 | 15.08341 | 16.74618 |
| | 20 | 9.23520 | 11.06896 | 12.83054 | 15.08371 | 16.74654 |
| | 25 | 9.23564 | 11.06954 | 12.83128 | 15.08467 | 16.74769 |
| | 30 | 9.23587 | 11.06984 | 12.83167 | 15.08518 | 16.74830 |
| | 35 | 9.23600 | 11.07002 | 12.83190 | 15.08548 | 16.74866 |
| | 40 | 9.23609 | 11.07014 | 12.83204 | 15.08567 | 16.74889 |
| | 45 | 9.23615 | 11.07022 | 12.83214 | 15.08580 | 16.74904 |
| | 50 | 9.23619 | 11.07027 | 12.83221 | 15.08589 | 16.74915 |
| | 60 | 9.23624 | 11.07034 | 12.83230 | 15.08601 | 16.74929 |
| | 70 | 9.23627 | 11.07038 | 12.83236 | 15.08608 | 16.74938 |
| | 80 | 9.23629 | 11.07041 | 12.83239 | 15.08613 | 16.74943 |
| | 90 | 9.23631 | 11.07043 | 12.83242 | 15.08616 | 16.74947 |
| | 100 | 9.23632 | 11.07044 | 12.83243 | 15.08618 | 16.74949 |
| $L_1$ | 10 | .916347 | .900594 | .885717 | .867048 | .853525 |
| | 11 | .924539 | .910251 | .896734 | .879738 | .867403 |
| | 12 | .931271 | .918200 | .905816 | .890220 | .878884 |
| | 13 | .936902 | .924856 | .913431 | .899024 | .888539 |
| | 14 | .941680 | .930512 | .919909 | .906523 | .896770 |
| | 15 | .945785 | .935376 | .925485 | .912986 | .903870 |
| | 16 | .949351 | .939604 | .930335 | .918613 | .910058 |
| | 17 | .952477 | .943313 | .934594 | .923558 | .915497 |
| | 18 | .955240 | .946594 | .938361 | .927937 | .920317 |
| | 19 | .957699 | .949515 | .941719 | .931841 | .924617 |
| | 20 | .959902 | .952134 | .944730 | .935344 | .928478 |
| | 25 | .968187 | .961991 | .956077 | .948565 | .943059 |
| | 30 | .973635 | .968483 | .963559 | .957298 | .952703 |
| | 35 | .977490 | .973081 | .968863 | .963496 | .959554 |
| | 40 | .980361 | .976508 | .972820 | .968123 | .964672 |
| | 45 | .982583 | .979161 | .975884 | .971710 | .968640 |
| | 50 | .984353 | .981276 | .978328 | .974570 | .971807 |
| | 60 | .986997 | .984435 | .981980 | .978849 | .976544 |
| | 70 | .988876 | .986682 | .984578 | .981895 | .979919 |
| | 80 | .990281 | .988362 | .986522 | .984174 | .982444 |
| | 90 | .991370 | .989666 | .988031 | .985943 | .984406 |
| | 100 | .992240 | .990707 | .989235 | .987357 | .985973 |

Table A15 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 7

| | N | $\alpha$=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 10.63876 | 12.58389 | 14.43974 | 16.79948 | 18.53292 |
| | 11 | 10.63990 | 12.58539 | 14.44161 | 16.80190 | 18.53577 |
| | 12 | 10.64074 | 12.58649 | 14.44299 | 16.80367 | 18.53787 |
| | 13 | 10.64138 | 12.58732 | 14.44403 | 16.80501 | 18.53945 |
| | 14 | 10.64187 | 12.58796 | 14.44484 | 16.80605 | 18.54068 |
| | 15 | 10.64226 | 12.58847 | 14.44547 | 16.80687 | 18.54165 |
| | 16 | 10.64257 | 12.58888 | 14.44598 | 16.80753 | 18.54243 |
| | 17 | 10.64282 | 12.58921 | 14.44640 | 16.80806 | 18.54306 |
| | 18 | 10.64303 | 12.58949 | 14.44675 | 16.80851 | 18.54359 |
| | 19 | 10.64321 | 12.58972 | 14.44703 | 16.80888 | 18.54402 |
| | 20 | 10.64336 | 12.58991 | 14.44728 | 16.80919 | 18.54439 |
| | 25 | 10.64384 | 12.59054 | 14.44807 | 16.81021 | 18.54560 |
| | 30 | 10.64410 | 12.59088 | 14.44848 | 16.81075 | 18.54623 |
| | 35 | 10.64425 | 12.59107 | 14.44873 | 16.81106 | 18.54660 |
| | 40 | 10.64434 | 12.59120 | 14.44889 | 16.81126 | 18.54684 |
| | 45 | 10.64441 | 12.59128 | 14.44899 | 16.81140 | 18.54700 |
| | 50 | 10.64445 | 12.59134 | 14.44907 | 16.81150 | 18.54711 |
| | 60 | 10.64451 | 12.59142 | 14.44916 | 16.81162 | 18.54726 |
| | 70 | 10.64455 | 12.59146 | 14.44922 | 16.81169 | 18.54735 |
| | 80 | 10.64457 | 12.59149 | 14.44926 | 16.81174 | 18.54740 |
| | 90 | 10.64458 | 12.59151 | 14.44928 | 16.81177 | 18.54744 |
| | 100 | 10.64460 | 12.59153 | 14.44930 | 16.81180 | 18.54747 |

| | N | $\alpha$=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $L_1$ | 10 | .917356 | .903001 | .889515 | .872658 | .860478 |
| | 11 | .925450 | .912433 | .900183 | .884844 | .873743 |
| | 12 | .932102 | .920194 | .908975 | .894905 | .884709 |
| | 13 | .937664 | .926693 | .916344 | .903352 | .893925 |
| | 14 | .942385 | .932214 | .922611 | .910543 | .901778 |
| | 15 | .946442 | .936962 | .928005 | .916739 | .908550 |
| | 16 | .949964 | .941089 | .932697 | .922134 | .914450 |
| | 17 | .953053 | .944708 | .936815 | .926872 | .919635 |
| | 18 | .955782 | .947909 | .940458 | .931067 | .924227 |
| | 19 | .958212 | .950760 | .943704 | .934807 | .928324 |
| | 20 | .960388 | .953315 | .946615 | .938162 | .932001 |
| | 25 | .968573 | .962933 | .957582 | .950820 | .945883 |
| | 30 | .973955 | .969265 | .964811 | .959177 | .955059 |
| | 35 | .977763 | .973750 | .969936 | .965107 | .961575 |
| | 40 | .980600 | .977093 | .973757 | .969533 | .966441 |
| | 45 | .982795 | .979680 | .976717 | .972962 | .970213 |
| | 50 | .984544 | .981742 | .979077 | .975698 | .973223 |
| | 60 | .987155 | .984823 | .982604 | .979788 | .977725 |
| | 70 | .989011 | .987014 | .985113 | .982700 | .980931 |
| | 80 | .990399 | .988653 | .986989 | .984878 | .983330 |
| | 90 | .991475 | .989924 | .988446 | .986570 | .985193 |
| | 100 | .992335 | .990939 | .989609 | .987920 | .986682 |

Table A16 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 8

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 12.01058 | 14.05884 | 16.00250 | 18.46228 | 20.26249 |
| | 11 | 12.01184 | 14.06045 | 16.00450 | 18.46482 | 20.26546 |
| | 12 | 12.01276 | 14.06164 | 16.00597 | 18.46668 | 20.26764 |
| | 13 | 12.01346 | 14.06254 | 16.00707 | 18.46809 | 20.26929 |
| | 14 | 12.01400 | 14.06323 | 16.00793 | 18.46918 | 20.27056 |
| | 15 | 12.01442 | 14.06378 | 16.00861 | 18.47004 | 20.27157 |
| | 16 | 12.01476 | 14.06422 | 16.00915 | 18.47073 | 20.27238 |
| | 17 | 12.01504 | 14.06458 | 16.00960 | 18.47129 | 20.27304 |
| | 18 | 12.01528 | 14.06488 | 16.00997 | 18.47176 | 20.27358 |
| | 19 | 12.01547 | 14.06512 | 16.01027 | 18.47215 | 20.27404 |
| | 20 | 12.01563 | 14.06533 | 16.01053 | 18.47247 | 20.27443 |
| | 25 | 12.01616 | 14.06601 | 16.01137 | 18.47354 | 20.27568 |
| | 30 | 12.01644 | 14.06637 | 16.01182 | 18.47410 | 20.27633 |
| | 35 | 12.01660 | 14.06658 | 16.01208 | 18.47443 | 20.27672 |
| | 40 | 12.01671 | 14.06672 | 16.01224 | 18.47465 | 20.27697 |
| | 45 | 12.01678 | 14.06681 | 16.01236 | 18.47479 | 20.27713 |
| | 50 | 12.01683 | 14.06687 | 16.01244 | 18.47489 | 20.27725 |
| | 60 | 12.01689 | 14.06696 | 16.01254 | 18.47502 | 20.27740 |
| | 70 | 12.01693 | 14.06701 | 16.01260 | 18.47510 | 20.27749 |
| | 80 | 12.01696 | 14.06704 | 16.01264 | 18.47515 | 20.27755 |
| | 90 | 12.01697 | 14.06706 | 16.01267 | 18.47518 | 20.27759 |
| | 100 | 12.01699 | 14.06708 | 16.01268 | 18.47521 | 20.27762 |
| $L_1$ | 10 | .918346 | .905102 | .892711 | .877272 | .866143 |
| | 11 | .926345 | .914336 | .903084 | .889042 | .878904 |
| | 12 | .932918 | .921935 | .911631 | .898756 | .889448 |
| | 13 | .938415 | .928296 | .918794 | .906908 | .898306 |
| | 14 | .943079 | .933699 | .924884 | .913846 | .905851 |
| | 15 | .947087 | .938346 | .930124 | .919822 | .912355 |
| | 16 | .950568 | .942384 | .934682 | .925024 | .918019 |
| | 17 | .953620 | .945926 | .938682 | .929592 | .922996 |
| | 18 | .956316 | .949058 | .942220 | .933636 | .927404 |
| | 19 | .958717 | .951847 | .945372 | .937241 | .931334 |
| | 20 | .960867 | .954346 | .948199 | .940475 | .934862 |
| | 25 | .968954 | .963755 | .958846 | .952670 | .948174 |
| | 30 | .974271 | .969948 | .965864 | .960718 | .956970 |
| | 35 | .978033 | .974334 | .970837 | .966428 | .963214 |
| | 40 | .980835 | .977603 | .974545 | .970688 | .967875 |
| | 45 | .983004 | .980133 | .977417 | .973989 | .971488 |
| | 50 | .984731 | .982150 | .979706 | .976622 | .974371 |
| | 60 | .987311 | .985162 | .983128 | .980558 | .978682 |
| | 70 | .989145 | .987305 | .985562 | .983360 | .981751 |
| | 80 | .990516 | .988907 | .987382 | .985456 | .984048 |
| | 90 | .991579 | .990149 | .988795 | .987083 | .985831 |
| | 100 | .992428 | .991142 | .989923 | .988382 | .987256 |

Table A17 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 9

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 13.35456 | 15.49842 | 17.52368 | 20.07661 | 21.93912 |
| | 11 | 13.35592 | 15.50015 | 17.52579 | 20.07926 | 21.94221 |
| | 12 | 13.35692 | 15.50142 | 17.52735 | 20.08121 | 21.94447 |
| | 13 | 13.35768 | 15.50238 | 17.52852 | 20.08268 | 21.94618 |
| | 14 | 13.35827 | 15.50313 | 17.52943 | 20.08382 | 21.94750 |
| | 15 | 13.35873 | 15.50371 | 17.53015 | 20.08472 | 21.94855 |
| | 16 | 13.35910 | 15.50419 | 17.53072 | 20.08544 | 21.94939 |
| | 17 | 13.35940 | 15.50457 | 17.53120 | 20.08603 | 21.95008 |
| | 18 | 13.35965 | 15.50489 | 17.53158 | 20.08652 | 21.95064 |
| | 19 | 13.35986 | 15.50515 | 17.53191 | 20.08693 | 21.95112 |
| | 20 | 13.36004 | 15.50538 | 17.53218 | 20.08727 | 21.95151 |
| | 25 | 13.36062 | 15.50611 | 17.53307 | 20.08839 | 21.95281 |
| | 30 | 13.36092 | 15.50649 | 17.53354 | 20.08898 | 21.95349 |
| | 35 | 13.36110 | 15.50672 | 17.53382 | 20.08932 | 21.95390 |
| | 40 | 13.36121 | 15.50686 | 17.53399 | 20.08954 | 21.95415 |
| | 45 | 13.36129 | 15.50696 | 17.53411 | 20.08969 | 21.95433 |
| | 50 | 13.36134 | 15.50703 | 17.53420 | 20.08980 | 21.95445 |
| | 60 | 13.36141 | 15.50712 | 17.53431 | 20.08993 | 21.95461 |
| | 70 | 13.36145 | 15.50717 | 17.53437 | 20.09002 | 21.95470 |
| | 80 | 13.36148 | 15.50720 | 17.53441 | 20.09007 | 21.95476 |
| | 90 | 13.36150 | 15.50723 | 17.53444 | 20.09010 | 21.95480 |
| | 100 | 13.36151 | 15.50724 | 17.53446 | 20.09013 | 21.95483 |
| $L_1$ | 10 | .919277 | .906940 | .895437 | .881145 | .870862 |
| | 11 | .927187 | .916002 | .905559 | .892565 | .883202 |
| | 12 | .933687 | .923457 | .913897 | .901985 | .893393 |
| | 13 | .939122 | .929698 | .920883 | .909889 | .901951 |
| | 14 | .943733 | .934999 | .926821 | .916614 | .909238 |
| | 15 | .947696 | .939557 | .931931 | .922406 | .915518 |
| | 16 | .951138 | .943517 | .936374 | .927446 | .920986 |
| | 17 | .954154 | .946991 | .940273 | .931872 | .925790 |
| | 18 | .956820 | .950062 | .943722 | .935789 | .930043 |
| | 19 | .959193 | .952798 | .946794 | .939281 | .933836 |
| | 20 | .961319 | .955249 | .949549 | .942412 | .937239 |
| | 25 | .969313 | .964474 | .959924 | .954219 | .950077 |
| | 30 | .974569 | .970546 | .966760 | .962009 | .958557 |
| | 35 | .978288 | .974845 | .971604 | .967533 | .964574 |
| | 40 | .981058 | .978050 | .975216 | .971655 | .969066 |
| | 45 | .983201 | .980530 | .978013 | .974849 | .972547 |
| | 50 | .984909 | .982507 | .980242 | .977395 | .975324 |
| | 60 | .987458 | .985459 | .983574 | .981202 | .979476 |
| | 70 | .989271 | .987559 | .985944 | .983912 | .982432 |
| | 80 | .990626 | .989129 | .987716 | .985939 | .984644 |
| | 90 | .991677 | .990347 | .989092 | .987512 | .986361 |
| | 100 | .992516 | .991319 | .990190 | .988769 | .987732 |

Table A18 Percentage Points of $L_1$ and $M_1 = -2u \log \lambda_1$

P = 10

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_1$ | 10 | 14.67611 | 16.90951 | 19.01131 | 21.65177 | 23.57293 |
| | 11 | 14.67758 | 16.91135 | 19.01354 | 21.65454 | 23.57613 |
| | 12 | 14.67865 | 16.91271 | 19.01518 | 21.65657 | 23.57848 |
| | 13 | 14.67947 | 16.91373 | 19.01641 | 21.65811 | 23.58025 |
| | 14 | 14.68010 | 16.91452 | 19.01737 | 21.65930 | 23.58163 |
| | 15 | 14.68060 | 16.91515 | 19.01813 | 21.66024 | 23.58271 |
| | 16 | 14.68100 | 16.91565 | 19.01874 | 21.66100 | 23.58358 |
| | 17 | 14.68133 | 16.91606 | 19.01923 | 21.66161 | 23.58429 |
| | 18 | 14.68160 | 16.91640 | 19.01964 | 21.66212 | 23.58488 |
| | 19 | 14.68182 | 16.91668 | 19.01999 | 21.66254 | 23.58537 |
| | 20 | 14.68201 | 16.91692 | 19.02028 | 21.66290 | 23.58578 |
| | 25 | 14.68263 | 16.91770 | 19.02122 | 21.66407 | 23.58713 |
| | 30 | 14.68296 | 16.91810 | 19.02171 | 21.66468 | 23.58784 |
| | 35 | 14.68315 | 16.91834 | 19.02200 | 21.66504 | 23.58825 |
| | 40 | 14.68327 | 16.91850 | 19.02219 | 21.66527 | 23.58852 |
| | 45 | 14.68336 | 16.91860 | 19.02231 | 21.66543 | 23.58870 |
| | 50 | 14.68341 | 16.91867 | 19.02240 | 21.66554 | 23.58883 |
| | 60 | 14.68349 | 16.91877 | 19.02252 | 21.66568 | 23.58899 |
| | 70 | 14.68354 | 16.91883 | 19.02258 | 21.66577 | 23.58909 |
| | 80 | 14.68356 | 16.91886 | 19.02263 | 21.66582 | 23.58915 |
| | 90 | 14.68358 | 16.91889 | 19.02266 | 21.66586 | 23.58919 |
| | 100 | 14.68360 | 16.91890 | 19.02268 | 21.66588 | 23.58922 |
| $L_1$ | 10 | .920140 | .908559 | .897794 | .884450 | .874867 |
| | 11 | .927968 | .917470 | .907698 | .895570 | .886848 |
| | 12 | .934399 | .924799 | .915855 | .904740 | .896738 |
| | 13 | .939776 | .930934 | .922688 | .912432 | .905041 |
| | 14 | .944340 | .936144 | .928495 | .918975 | .912109 |
| | 15 | .948260 | .940623 | .933492 | .924609 | .918199 |
| | 16 | .951665 | .944516 | .937836 | .929511 | .923500 |
| | 17 | .954650 | .947929 | .941648 | .933815 | .928156 |
| | 18 | .957287 | .950947 | .945019 | .937624 | .932279 |
| | 19 | .959635 | .953635 | .948023 | .941018 | .935954 |
| | 20 | .961738 | .956043 | .950715 | .944062 | .939251 |
| | 25 | .969646 | .965107 | .960854 | .955538 | .951688 |
| | 30 | .974845 | .971072 | .967534 | .963107 | .959899 |
| | 35 | .978524 | .975296 | .972267 | .968475 | .965725 |
| | 40 | .981264 | .978443 | .975795 | .972479 | .970072 |
| | 45 | .983384 | .980879 | .978527 | .975580 | .973442 |
| | 50 | .985073 | .982821 | .980705 | .978054 | .976129 |
| | 60 | .987595 | .985720 | .983959 | .981751 | .980147 |
| | 70 | .989388 | .987783 | .986274 | .984382 | .983007 |
| | 80 | .990728 | .989324 | .988005 | .986350 | .985147 |
| | 90 | .991768 | .990521 | .989348 | .987877 | .986808 |
| | 100 | .992598 | .991476 | .990421 | .989097 | .988135 |

Table A19 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 2

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 11 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 12 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 13 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 14 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 15 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 16 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 17 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 18 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 19 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 20 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 25 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 30 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 35 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 40 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 45 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 50 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 60 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 70 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 80 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 90 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| | 100 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.59664 |
| $L_2$ | 10 | .879923 | .846682 | .814698 | .774264 | .745015 |
| | 11 | .891251 | .860892 | .831567 | .794328 | .767270 |
| | 12 | .900628 | .872695 | .845627 | .811131 | .785973 |
| | 13 | .908518 | .882654 | .857526 | .825404 | .801907 |
| | 14 | .915247 | .891170 | .867725 | .837678 | .815641 |
| | 15 | .921055 | .898534 | .876564 | .848343 | .827600 |
| | 16 | .926119 | .904966 | .884297 | .857696 | .838106 |
| | 17 | .930572 | .910632 | .891119 | .865964 | .847408 |
| | 18 | .934519 | .915660 | .897182 | .873326 | .855702 |
| | 19 | .938042 | .920153 | .902606 | .879923 | .863142 |
| | 20 | .941205 | .924192 | .907487 | .885867 | .869854 |
| | 25 | .953162 | .939497 | .926027 | .908518 | .895492 |
| | 30 | .961078 | .949661 | .938379 | .923671 | .912698 |
| | 35 | .966705 | .956901 | .947197 | .934519 | .925042 |
| | 40 | .970911 | .962321 | .953808 | .942668 | .934329 |
| | 45 | .974174 | .966531 | .958947 | .949014 | .941569 |
| | 50 | .976778 | .969894 | .963058 | .954095 | .947371 |
| | 60 | .980676 | .974932 | .969222 | .961725 | .956092 |
| | 70 | .983453 | .978526 | .973623 | .967180 | .962334 |
| | 80 | .985532 | .981218 | .976923 | .971274 | .967022 |
| | 90 | .987147 | .983311 | .979489 | .974460 | .970673 |
| | 100 | .988438 | .984984 | .981542 | .977010 | .973596 |

AD-A151 906 ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTION OF TEST STATISTICS ASSOCIATED W..(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI.. D J LAWTON 2/2

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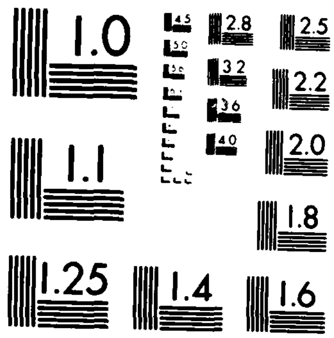
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2.8
2.5
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1.1
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1.25
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1.6

Table A20 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 3

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 7.78239 | 9.49178 | 11.14855 | 13.28376 | 14.86880 |
| | 11 | 7.78184 | 9.49102 | 11.14757 | 13.28244 | 14.86720 |
| | 12 | 7.78143 | 9.49046 | 11.14683 | 13.28146 | 14.86601 |
| | 13 | 7.78112 | 9.49003 | 11.14628 | 13.28071 | 14.86511 |
| | 14 | 7.78087 | 9.48969 | 11.14584 | 13.28012 | 14.86440 |
| | 15 | 7.78068 | 9.48942 | 11.14549 | 13.27966 | 14.86384 |
| | 16 | 7.78052 | 9.48921 | 11.14521 | 13.27928 | 14.86338 |
| | 17 | 7.78039 | 9.48903 | 11.14498 | 13.27897 | 14.86301 |
| | 18 | 7.78028 | 9.48888 | 11.14479 | 13.27872 | 14.86269 |
| | 19 | 7.78019 | 9.48876 | 11.14463 | 13.27850 | 14.86243 |
| | 20 | 7.78012 | 9.48866 | 11.14449 | 13.27832 | 14.86221 |
| | 25 | 7.77987 | 9.48831 | 11.14404 | 13.27772 | 14.85149 |
| | 30 | 7.77973 | 9.48813 | 11.14381 | 13.27740 | 14.86110 |
| | 35 | 7.77965 | 9.48802 | 11.14367 | 13.27721 | 14.86087 |
| | 40 | 7.77960 | 9.48795 | 11.14358 | 13.27709 | 14.86073 |
| | 45 | 7.77957 | 9.48790 | 11.14351 | 13.27701 | 14.86063 |
| | 50 | 7.77954 | 9.48787 | 11.14347 | 13.27695 | 14.86056 |
| | 60 | 7.77951 | 9.48783 | 11.14341 | 13.27687 | 14.86046 |
| | 70 | 7.77949 | 9.48780 | 11.14338 | 13.27683 | 14.86041 |
| | 80 | 7.77948 | 9.48778 | 11.14336 | 13.27680 | 14.86037 |
| | 90 | 7.77947 | 9.48777 | 11.14334 | 13.27678 | 14.86035 |
| | 100 | 7.77947 | 9.48776 | 11.14333 | 13.27676 | 14.86033 |
| $L_2$ | 10 | .868057 | .841493 | .816522 | .785431 | .763119 |
| | 11 | .880231 | .855910 | .832979 | .804331 | .783703 |
| | 12 | .890350 | .867926 | .846732 | .820181 | .801011 |
| | 13 | .898894 | .878095 | .858397 | .833662 | .815763 |
| | 14 | .906203 | .886812 | .868414 | .845267 | .828485 |
| | 15 | .912527 | .894366 | .877110 | .855361 | .839567 |
| | 16 | .918053 | .900976 | .884728 | .864222 | .849308 |
| | 17 | .922922 | .906808 | .891458 | .872060 | .857935 |
| | 18 | .927246 | .911991 | .897446 | .879045 | .865630 |
| | 19 | .931110 | .916629 | .902809 | .885307 | .872535 |
| | 20 | .934585 | .920802 | .907639 | .890953 | .878766 |
| | 25 | .947760 | .936658 | .926024 | .912497 | .902584 |
| | 30 | .956518 | .947226 | .938306 | .926936 | .918585 |
| | 35 | .962762 | .954772 | .947092 | .937286 | .930073 |
| | 40 | .967437 | .960430 | .953687 | .945068 | .938721 |
| | 45 | .971070 | .964830 | .958821 | .951132 | .945466 |
| | 50 | .973973 | .968349 | .962930 | .955991 | .950873 |
| | 60 | .978324 | .973628 | .969098 | .963291 | .959004 |
| | 70 | .981429 | .977397 | .973506 | .968514 | .964825 |
| | 80 | .983756 | .980224 | .976813 | .972436 | .969199 |
| | 90 | .985564 | .982422 | .979387 | .975489 | .972606 |
| | 100 | .987010 | .984181 | .981446 | .977933 | .975334 |

Table A21 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 4

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 10.65069 | 12.59949 | 14.45926 | 16.82460 | 18.56258 |
| | 11 | 10.64956 | 12.59802 | 14.45742 | 16.82224 | 18.55980 |
| | 12 | 10.64873 | 12.59693 | 14.45605 | 16.82048 | 18.55772 |
| | 13 | 10.64809 | 12.59609 | 14.45501 | 16.81914 | 18.55613 |
| | 14 | 10.64759 | 12.59544 | 14.45419 | 16.81808 | 18.55489 |
| | 15 | 10.64719 | 12.59491 | 14.45354 | 16.81725 | 18.55390 |
| | 16 | 10.64686 | 12.59449 | 14.45301 | 16.81657 | 18.55310 |
| | 17 | 10.64660 | 12.59415 | 14.45258 | 16.81601 | 18.55244 |
| | 18 | 10.64638 | 12.59386 | 14.45222 | 16.81555 | 18.55189 |
| | 19 | 10.64619 | 12.59362 | 14.45191 | 16.81516 | 18.55144 |
| | 20 | 10.64604 | 12.59341 | 14.45166 | 16.81483 | 18.55105 |
| | 25 | 10.64552 | 12.59274 | 14.45081 | 16.81374 | 18.54977 |
| | 30 | 10.64525 | 12.59238 | 14.45036 | 16.81316 | 18.54908 |
| | 35 | 10.64508 | 12.59216 | 14.45010 | 16.81282 | 18.54868 |
| | 40 | 10.64498 | 12.59203 | 14.44992 | 16.81260 | 18.54842 |
| | 45 | 10.64491 | 12.59193 | 14.44981 | 16.81245 | 18.54824 |
| | 50 | 10.64485 | 12.59187 | 14.44972 | 16.81234 | 18.54811 |
| | 60 | 10.64479 | 12.59178 | 14.44962 | 16.81220 | 18.54795 |
| | 70 | 10.64475 | 12.59173 | 14.44955 | 16.81212 | 18.54785 |
| | 80 | 10.64472 | 12.59170 | 14.44951 | 16.81207 | 18.54779 |
| | 90 | 10.64471 | 12.59167 | 14.44948 | 16.81203 | 18.54775 |
| | 100 | 10.64469 | 12.59166 | 14.44946 | 16.81200 | 18.54771 |
| $L_2$ | 10 | .865950 | .843443 | .822509 | .796634 | .778142 |
| | 11 | .878207 | .857586 | .838358 | .814526 | .797448 |
| | 12 | .888412 | .869388 | .851613 | .829531 | .813673 |
| | 13 | .897041 | .879386 | .862862 | .842296 | .827498 |
| | 14 | .904431 | .887963 | .872528 | .853285 | .839418 |
| | 15 | .910832 | .895403 | .880923 | .862845 | .849801 |
| | 16 | .916430 | .901916 | .888281 | .871237 | .858924 |
| | 17 | .921367 | .907667 | .894783 | .878663 | .867005 |
| | 18 | .925753 | .912781 | .900571 | .885280 | .874211 |
| | 19 | .929676 | .917358 | .905756 | .891213 | .880677 |
| | 20 | .933205 | .921479 | .910427 | .896563 | .886511 |
| | 25 | .946605 | .937151 | .928218 | .916981 | .908812 |
| | 30 | .955527 | .947609 | .940114 | .930669 | .923791 |
| | 35 | .961894 | .955083 | .948629 | .940484 | .934545 |
| | 40 | .966667 | .960691 | .955024 | .947865 | .942639 |
| | 45 | .970377 | .965055 | .960003 | .953617 | .948953 |
| | 50 | .973344 | .968546 | .963990 | .958227 | .954014 |
| | 60 | .977793 | .973785 | .969976 | .965153 | .961625 |
| | 70 | .980969 | .977528 | .974255 | .970109 | .967074 |
| | 80 | .983350 | .980335 | .977467 | .973831 | .971169 |
| | 90 | .985202 | .982519 | .979966 | .976729 | .974357 |
| | 100 | .986683 | .984267 | .981966 | .979049 | .976911 |

Table A22 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 5

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 13.37060 | 15.51876 | 17.54852 | 20.10773 | 21.97526 |
| | 11 | 13.36893 | 15.51664 | 17.54593 | 20.10449 | 21.97150 |
| | 12 | 13.36768 | 15.51506 | 17.54401 | 20.10208 | 21.96870 |
| | 13 | 13.36673 | 15.51385 | 17.54253 | 20.10023 | 21.96656 |
| | 14 | 13.36598 | 15.51291 | 17.54137 | 20.09878 | 21.96488 |
| | 15 | 13.36538 | 15.51215 | 17.54045 | 20.09763 | 21.96354 |
| | 16 | 13.36490 | 15.51154 | 17.53970 | 20.09669 | 21.96245 |
| | 17 | 13.36450 | 15.51104 | 17.53909 | 20.09593 | 21.96156 |
| | 18 | 13.36417 | 15.51062 | 17.53858 | 20.09529 | 21.96082 |
| | 19 | 13.36390 | 15.51027 | 17.53815 | 20.09475 | 21.96019 |
| | 20 | 13.36366 | 15.50997 | 17.53779 | 20.09429 | 21.95967 |
| | 25 | 13.36289 | 15.50899 | 17.53659 | 20.09280 | 21.95793 |
| | 30 | 13.36248 | 15.50847 | 17.53595 | 20.09200 | 21.95700 |
| | 35 | 13.36223 | 15.50815 | 17.53557 | 20.09152 | 21.95645 |
| | 40 | 13.36207 | 15.50795 | 17.53533 | 20.09121 | 21.95609 |
| | 45 | 13.36196 | 15.50782 | 17.53516 | 20.09101 | 21.95585 |
| | 50 | 13.36189 | 15.50772 | 17.53504 | 20.09086 | 21.95568 |
| | 60 | 13.36179 | 15.50759 | 17.53489 | 20.09067 | 21.95545 |
| | 70 | 13.36173 | 15.50752 | 17.53480 | 20.09055 | 21.95532 |
| | 80 | 13.36169 | 15.50747 | 17.53474 | 20.09048 | 21.95523 |
| | 90 | 13.36166 | 15.50744 | 17.53470 | 20.09043 | 21.95518 |
| | 100 | 13.36165 | 15.50741 | 17.53467 | 20.09039 | 21.95513 |
| $L_2$ | 10 | .866087 | .846311 | .828040 | .805564 | .789549 |
| | 11 | .878275 | .860151 | .843371 | .822679 | .807901 |
| | 12 | .888431 | .871707 | .856195 | .837031 | .823318 |
| | 13 | .897024 | .881501 | .867082 | .849237 | .836449 |
| | 14 | .904389 | .889907 | .876437 | .859746 | .847767 |
| | 15 | .910771 | .897200 | .884564 | .868886 | .857623 |
| | 16 | .916354 | .903587 | .891688 | .876910 | .866281 |
| | 17 | .921280 | .909228 | .897985 | .884009 | .873949 |
| | 18 | .925659 | .914245 | .903590 | .890335 | .880786 |
| | 19 | .929576 | .918737 | .908612 | .896007 | .886919 |
| | 20 | .933101 | .922782 | .913137 | .901121 | .892454 |
| | 25 | .946493 | .938171 | .930375 | .920640 | .913601 |
| | 30 | .955419 | .948446 | .941906 | .933725 | .927800 |
| | 35 | .961792 | .955794 | .950161 | .943107 | .937993 |
| | 40 | .966571 | .961308 | .956362 | .950162 | .945664 |
| | 45 | .970288 | .965599 | .961190 | .955661 | .951646 |
| | 50 | .973261 | .969034 | .965057 | .960067 | .956442 |
| | 60 | .977719 | .974188 | .970863 | .966688 | .963653 |
| | 70 | .980904 | .977871 | .975015 | .971426 | .968815 |
| | 80 | .983292 | .980635 | .978131 | .974983 | .972693 |
| | 90 | .985149 | .982784 | .980556 | .977754 | .975714 |
| | 100 | .986634 | .984505 | .982497 | .979971 | .978133 |

Table A23 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 6

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 15.99908 | 18.32180 | 20.50088 | 23.23099 | 25.21310 |
| | 11 | 15.99689 | 18.31908 | 20.49762 | 23.22699 | 25.20851 |
| | 12 | 15.99525 | 18.31705 | 20.49518 | 23.22399 | 25.20508 |
| | 13 | 15.99399 | 18.31549 | 20.49331 | 23.22170 | 25.20245 |
| | 14 | 15.99301 | 18.31427 | 20.49185 | 23.21990 | 25.20038 |
| | 15 | 15.99222 | 18.31329 | 20.49068 | 23.21846 | 25.19874 |
| | 16 | 15.99159 | 18.31250 | 20.48973 | 23.21730 | 25.19741 |
| | 17 | 15.99106 | 18.31185 | 20.48895 | 23.21635 | 25.19631 |
| | 18 | 15.99063 | 18.31131 | 20.48831 | 23.21555 | 25.19540 |
| | 19 | 15.99026 | 18.31086 | 20.48776 | 23.21488 | 25.19463 |
| | 20 | 15.98995 | 18.31048 | 20.48730 | 23.21431 | 25.19398 |
| | 25 | 15.98893 | 18.30921 | 20.48578 | 23.21245 | 25.19184 |
| | 30 | 15.98838 | 18.30853 | 20.48497 | 23.21145 | 25.19070 |
| | 35 | 15.98806 | 18.30813 | 20.48448 | 23.21086 | 25.19002 |
| | 40 | 15.98785 | 18.30787 | 20.48417 | 23.21047 | 25.18958 |
| | 45 | 15.98771 | 18.30769 | 20.48396 | 23.21021 | 25.18928 |
| | 50 | 15.98761 | 18.30757 | 20.48381 | 23.21003 | 25.18907 |
| | 60 | 15.98747 | 18.30740 | 20.48362 | 23.20979 | 25.18880 |
| | 70 | 15.98740 | 18.30731 | 20.48350 | 23.20965 | 25.18863 |
| | 80 | 15.98734 | 18.30724 | 20.48342 | 23.20955 | 25.18852 |
| | 90 | 15.98731 | 18.30720 | 20.48337 | 23.20949 | 25.18845 |
| | 100 | 15.98728 | 18.30717 | 20.48333 | 23.20944 | 25.18840 |
| $L_2$ | 10 | .866885 | .849092 | .832732 | .812679 | .798423 |
| | 11 | .878968 | .862660 | .847637 | .829183 | .816038 |
| | 12 | .889041 | .873992 | .860106 | .843020 | .830829 |
| | 13 | .897567 | .883598 | .870691 | .854787 | .843423 |
| | 14 | .904877 | .891844 | .879788 | .864915 | .854275 |
| | 15 | .911214 | .899000 | .887691 | .873724 | .863722 |
| | 16 | .916759 | .905268 | .894619 | .881455 | .872020 |
| | 17 | .921652 | .910804 | .900743 | .888296 | .879367 |
| | 18 | .926003 | .915729 | .906194 | .894390 | .885917 |
| | 19 | .929896 | .920138 | .911079 | .899855 | .891793 |
| | 20 | .933399 | .924110 | .915479 | .904782 | .897094 |
| | 25 | .946716 | .939223 | .932248 | .923583 | .917343 |
| | 30 | .955595 | .949317 | .943465 | .936186 | .930936 |
| | 35 | .961937 | .956536 | .951496 | .945221 | .940691 |
| | 40 | .966695 | .961955 | .957530 | .952015 | .948031 |
| | 45 | .970395 | .966172 | .962228 | .957310 | .953755 |
| | 50 | .973355 | .969548 | .965991 | .961552 | .958343 |
| | 60 | .977796 | .974615 | .971641 | .967928 | .965241 |
| | 70 | .980968 | .978236 | .975681 | .972489 | .970179 |
| | 80 | .983347 | .980953 | .978714 | .975915 | .973888 |
| | 90 | .985197 | .983067 | .981074 | .978582 | .976777 |
| | 100 | .986677 | .984759 | .982963 | .980717 | .979090 |

Table A24 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 7

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 18.56403 | 21.04398 | 23.35785 | 26.24260 | 28.32863 |
| | 11 | 18.56133 | 21.04068 | 23.35395 | 26.23789 | 28.32327 |
| | 12 | 18.55931 | 21.03822 | 23.35104 | 26.23436 | 28.31927 |
| | 13 | 18.55776 | 21.03633 | 23.34880 | 26.23166 | 28.31620 |
| | 14 | 18.55655 | 21.03485 | 23.34705 | 26.22954 | 28.31379 |
| | 15 | 18.55558 | 21.03367 | 23.34566 | 26.22785 | 28.31187 |
| | 16 | 18.55479 | 21.03271 | 23.34452 | 26.22648 | 28.31032 |
| | 17 | 18.55415 | 21.03192 | 23.34359 | 26.22535 | 28.30904 |
| | 18 | 18.55361 | 21.03127 | 23.34282 | 26.22441 | 28.30797 |
| | 19 | 18.55316 | 21.03072 | 23.34216 | 26.22362 | 28.30707 |
| | 20 | 18.55277 | 21.03025 | 23.34161 | 26.22295 | 28.30631 |
| | 25 | 18.55151 | 21.02871 | 23.33979 | 26.22075 | 28.30381 |
| | 30 | 18.55084 | 21.02789 | 23.33881 | 26.21957 | 28.30247 |
| | 35 | 18.55044 | 21.02740 | 23.33823 | 26.21887 | 28.30168 |
| | 40 | 18.55018 | 21.02708 | 23.33786 | 26.21841 | 28.30116 |
| | 45 | 18.55000 | 21.02687 | 23.33761 | 26.21811 | 28.30081 |
| | 50 | 18.54988 | 21.02671 | 23.33743 | 26.21789 | 28.30056 |
| | 60 | 18.54971 | 21.02651 | 23.33719 | 26.21760 | 28.30024 |
| | 70 | 18.54962 | 21.02640 | 23.33705 | 26.21743 | 28.30005 |
| | 80 | 18.54955 | 21.02632 | 23.33696 | 26.21732 | 28.29992 |
| | 90 | 18.54951 | 21.02627 | 23.33690 | 26.21725 | 28.29984 |
| | 100 | 18.54948 | 21.02623 | 23.33685 | 26.21719 | 28.29978 |
| $L_2$ | 10 | .867873 | .851598 | .836688 | .818465 | .805535 |
| | 11 | .879845 | .864929 | .851240 | .834476 | .822561 |
| | 12 | .889830 | .876064 | .863413 | .847897 | .836852 |
| | 13 | .898283 | .885505 | .873747 | .859308 | .849016 |
| | 14 | .905532 | .893610 | .882629 | .869128 | .859495 |
| | 15 | .911817 | .900644 | .890344 | .877668 | .868615 |
| | 16 | .917318 | .906806 | .897108 | .885163 | .876625 |
| | 17 | .922173 | .912249 | .903086 | .891793 | .883715 |
| | 18 | .926490 | .917091 | .908409 | .897700 | .890036 |
| | 19 | .930353 | .921427 | .913177 | .902996 | .895705 |
| | 20 | .933830 | .925332 | .917474 | .907771 | .900819 |
| | 25 | .947050 | .940195 | .933845 | .925988 | .920349 |
| | 30 | .955868 | .950124 | .944797 | .938198 | .933455 |
| | 35 | .962168 | .957226 | .952638 | .946950 | .942858 |
| | 40 | .966894 | .962557 | .958529 | .953531 | .949934 |
| | 45 | .970571 | .966707 | .963117 | .958660 | .955450 |
| | 50 | .973512 | .970029 | .966791 | .962769 | .959871 |
| | 60 | .977925 | .975015 | .972307 | .968943 | .966518 |
| | 70 | .981078 | .978578 | .976252 | .973361 | .971275 |
| | 80 | .983442 | .981252 | .979214 | .976678 | .974849 |
| | 90 | .985282 | .983333 | .981518 | .979261 | .977632 |
| | 100 | .986753 | .984998 | .983363 | .981329 | .979861 |

Table A25 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 8

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 21.08152 | 23.70571 | 26.14343 | 29.17052 | 31.35233 |
| | 11 | 21.07833 | 23.70187 | 26.13893 | 29.16514 | 31.34628 |
| | 12 | 21.07594 | 23.69899 | 26.13557 | 29.16112 | 31.34174 |
| | 13 | 21.07411 | 23.69679 | 26.13299 | 29.15803 | 31.33827 |
| | 14 | 21.07267 | 23.69506 | 26.13097 | 29.15561 | 31.33554 |
| | 15 | 21.07153 | 23.69368 | 26.12935 | 29.15368 | 31.33337 |
| | 16 | 21.07060 | 23.69256 | 26.12804 | 29.15211 | 31.33160 |
| | 17 | 21.06983 | 23.69164 | 26.12697 | 29.15082 | 31.33015 |
| | 18 | 21.06920 | 23.69088 | 26.12607 | 29.14975 | 31.32894 |
| | 19 | 21.06866 | 23.69023 | 26.12531 | 29.14885 | 31.32793 |
| | 20 | 21.06821 | 23.68968 | 26.12467 | 29.14808 | 31.32706 |
| | 25 | 21.06671 | 23.68788 | 26.12257 | 29.14556 | 31.32422 |
| | 30 | 21.06591 | 23.68692 | 26.12144 | 29.14422 | 31.32271 |
| | 35 | 21.06544 | 23.68635 | 26.12077 | 29.14341 | 31.32180 |
| | 40 | 21.06513 | 23.68598 | 26.12033 | 29.14290 | 31.32122 |
| | 45 | 21.06492 | 23.68572 | 26.12004 | 29.14254 | 31.32082 |
| | 50 | 21.06477 | 23.68554 | 26.11983 | 29.14229 | 31.32054 |
| | 60 | 21.06458 | 23.68531 | 26.11956 | 29.14197 | 31.32017 |
| | 70 | 21.06446 | 23.68517 | 26.11939 | 29.14177 | 31.31995 |
| | 80 | 21.06439 | 23.68508 | 26.11929 | 29.14165 | 31.31981 |
| | 90 | 21.06434 | 23.68502 | 26.11922 | 29.14156 | 31.31971 |
| | 100 | 21.06430 | 23.68498 | 26.11917 | 29.14150 | 31.31964 |
| $L_2$ | 10 | .868886 | .853817 | .840054 | .823271 | .811383 |
| | 11 | .880753 | .866943 | .854308 | .838875 | .827925 |
| | 12 | .890652 | .877907 | .866232 | .851951 | .841805 |
| | 13 | .899034 | .887204 | .876354 | .863067 | .853616 |
| | 14 | .906222 | .895185 | .885053 | .872632 | .863788 |
| | 15 | .912456 | .902113 | .892610 | .880949 | .872640 |
| | 16 | .917913 | .908181 | .899234 | .888248 | .880413 |
| | 17 | .922729 | .913542 | .905090 | .894704 | .887292 |
| | 18 | .927012 | .918311 | .910302 | .900455 | .893424 |
| | 19 | .930845 | .922582 | .914972 | .905611 | .898923 |
| | 20 | .934296 | .926429 | .919180 | .910259 | .903883 |
| | 25 | .947416 | .941070 | .935213 | .927992 | .922822 |
| | 30 | .956169 | .950851 | .945939 | .939874 | .935528 |
| | 35 | .962423 | .957848 | .953618 | .948391 | .944643 |
| | 40 | .967116 | .963101 | .959387 | .954795 | .951499 |
| | 45 | .970767 | .967190 | .963880 | .959785 | .956845 |
| | 50 | .973688 | .970463 | .967478 | .963783 | .961129 |
| | 60 | .978071 | .975376 | .972880 | .969790 | .967569 |
| | 70 | .981202 | .978888 | .976744 | .974088 | .972178 |
| | 80 | .983551 | .981523 | .979644 | .977315 | .975640 |
| | 90 | .985378 | .983573 | .981901 | .979828 | .978336 |
| | 100 | .986839 | .985214 | .983707 | .981839 | .980495 |

Table A26 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 9

| | N | α=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 23.56183 | 26.32005 | 28.87299 | 32.03266 | 34.30383 |
| | 11 | 23.55816 | 26.31568 | 28.86792 | 32.02665 | 34.29710 |
| | 12 | 23.55542 | 26.31241 | 28.86413 | 32.02216 | 34.29208 |
| | 13 | 23.55331 | 26.30990 | 28.86122 | 32.01871 | 34.28822 |
| | 14 | 23.55166 | 26.30794 | 28.85893 | 32.01601 | 34.28519 |
| | 15 | 23.55034 | 26.30636 | 28.85711 | 32.01385 | 34.28277 |
| | 16 | 23.54927 | 26.30509 | 28.85563 | 32.01210 | 34.28081 |
| | 17 | 23.54839 | 26.30404 | 28.85441 | 32.01066 | 34.27920 |
| | 18 | 23.54765 | 26.30317 | 28.85340 | 32.00946 | 34.27786 |
| | 19 | 23.54704 | 26.30243 | 28.85255 | 32.00845 | 34.27673 |
| | 20 | 23.54651 | 26.30181 | 28.85183 | 32.00759 | 34.27577 |
| | 25 | 23.54479 | 26.29975 | 28.84944 | 32.00477 | 34.27261 |
| | 30 | 23.54387 | 26.29866 | 28.84817 | 32.00326 | 34.27092 |
| | 35 | 23.54332 | 26.29800 | 28.84741 | 32.00236 | 34.26991 |
| | 40 | 23.54296 | 26.29758 | 28.84692 | 32.00178 | 34.26927 |
| | 45 | 23.54272 | 26.29729 | 28.84659 | 32.00139 | 34.26882 |
| | 50 | 23.54255 | 26.29709 | 28.84635 | 32.00111 | 34.26851 |
| | 60 | 23.54233 | 26.29682 | 28.84604 | 32.00074 | 34.26810 |
| | 70 | 23.54220 | 26.29666 | 28.84586 | 32.00053 | 34.26786 |
| | 80 | 23.54211 | 26.29656 | 28.84574 | 32.00038 | 34.26770 |
| | 90 | 23.54205 | 26.29649 | 28.84566 | 32.00029 | 34.26759 |
| | 100 | 23.54201 | 26.29644 | 28.84560 | 32.00022 | 34.26751 |
| $L_2$ | 10 | .869863 | .855782 | .842951 | .827338 | .816294 |
| | 11 | .881633 | .868728 | .856952 | .842598 | .832429 |
| | 12 | .891451 | .879543 | .868662 | .855383 | .845964 |
| | 13 | .899766 | .888712 | .878603 | .866250 | .857479 |
| | 14 | .906899 | .896586 | .887145 | .875599 | .867394 |
| | 15 | .913083 | .903419 | .894565 | .883728 | .876020 |
| | 16 | .918498 | .909405 | .901070 | .890861 | .883594 |
| | 17 | .923278 | .914694 | .906820 | .897169 | .890297 |
| | 18 | .927528 | .919398 | .911938 | .902789 | .896270 |
| | 19 | .931332 | .923612 | .916523 | .907826 | .901626 |
| | 20 | .934757 | .927406 | .920655 | .912368 | .906457 |
| | 25 | .947780 | .941851 | .936397 | .929690 | .924900 |
| | 30 | .956470 | .951502 | .946927 | .941296 | .937270 |
| | 35 | .962680 | .958405 | .954466 | .949614 | .946141 |
| | 40 | .967339 | .963588 | .960130 | .955867 | .952815 |
| | 45 | .970965 | .967623 | .964541 | .960740 | .958017 |
| | 50 | .973866 | .970853 | .968073 | .964643 | .962186 |
| | 60 | .978218 | .975701 | .973377 | .970508 | .968452 |
| | 70 | .981328 | .979166 | .977169 | .974705 | .972937 |
| | 80 | .983660 | .981766 | .980016 | .977855 | .976305 |
| | 90 | .985475 | .983789 | .982232 | .980308 | .978928 |
| | 100 | .986927 | .985409 | .984005 | .982272 | .981028 |

Table A27 Percentage Points of $L_2$ and $M_2 = -2b \log \lambda_2$

P = 10

| | N | $\alpha$=.100 | .050 | .025 | .010 | .005 |
|---|---|---|---|---|---|---|
| $M_2$ | 10 | 26.01200 | 28.89595 | 31.55707 | 34.84134 | 37.19658 |
| | 11 | 26.00786 | 28.89107 | 31.55144 | 34.83474 | 37.18922 |
| | 12 | 26.00477 | 28.88741 | 31.54723 | 34.82980 | 37.18372 |
| | 13 | 26.00239 | 28.88461 | 31.54400 | 34.82600 | 37.17949 |
| | 14 | 26.00053 | 28.88240 | 31.54147 | 34.82303 | 37.17618 |
| | 15 | 25.99904 | 28.88065 | 31.53944 | 34.82065 | 37.17353 |
| | 16 | 25.99783 | 28.87922 | 31.53780 | 34.81872 | 37.17139 |
| | 17 | 25.99683 | 28.87805 | 31.53645 | 34.81713 | 37.16962 |
| | 18 | 25.99601 | 28.87707 | 31.53533 | 34.81581 | 37.16815 |
| | 19 | 25.99531 | 28.87625 | 31.53438 | 34.81470 | 37.16691 |
| | 20 | 25.99472 | 28.87555 | 31.53358 | 34.81376 | 37.16586 |
| | 25 | 25.99277 | 28.87325 | 31.53093 | 34.81065 | 37.16240 |
| | 30 | 25.99173 | 28.87202 | 31.52951 | 34.80898 | 37.16055 |
| | 35 | 25.99111 | 28.87129 | 31.52867 | 34.80799 | 37.15944 |
| | 40 | 25.99071 | 28.87081 | 31.52812 | 34.80735 | 37.15873 |
| | 45 | 25.99043 | 28.87049 | 31.52775 | 34.80692 | 37.15825 |
| | 50 | 25.99024 | 28.87026 | 31.52749 | 34.80661 | 37.15790 |
| | 60 | 25.98999 | 28.86997 | 31.52715 | 34.80621 | 37.15746 |
| | 70 | 25.98984 | 28.86979 | 31.52694 | 34.80597 | 37.15719 |
| | 80 | 25.98974 | 28.86967 | 31.52681 | 34.80581 | 37.15701 |
| | 90 | 25.98967 | 28.86959 | 31.52672 | 34.80570 | 37.15689 |
| | 100 | 25.98963 | 28.86954 | 31.52665 | 34.80563 | 37.15681 |
| $L_2$ | 10 | .870784 | .857528 | .845475 | .830833 | .820490 |
| | 11 | .882464 | .870316 | .859255 | .845798 | .836278 |
| | 12 | .892208 | .880999 | .870781 | .858334 | .849518 |
| | 13 | .900461 | .890057 | .880563 | .868987 | .860779 |
| | 14 | .907541 | .897834 | .888970 | .878151 | .870474 |
| | 15 | .913680 | .904584 | .896271 | .886118 | .878908 |
| | 16 | .919056 | .910498 | .902673 | .893108 | .886312 |
| | 17 | .923801 | .915722 | .908330 | .899290 | .892863 |
| | 18 | .928021 | .920369 | .913366 | .904796 | .898701 |
| | 19 | .931798 | .924532 | .917878 | .909732 | .903936 |
| | 20 | .935198 | .928280 | .921943 | .914181 | .908656 |
| | 25 | .948130 | .942550 | .937431 | .931151 | .926675 |
| | 30 | .956759 | .952084 | .947791 | .942519 | .938757 |
| | 35 | .962927 | .958904 | .955208 | .950666 | .947422 |
| | 40 | .967555 | .964025 | .960780 | .956790 | .953939 |
| | 45 | .971156 | .968011 | .965119 | .961561 | .959019 |
| | 50 | .974037 | .971202 | .968593 | .965384 | .963089 |
| | 60 | .978360 | .975992 | .973811 | .971127 | .969207 |
| | 70 | .981450 | .979415 | .977542 | .975235 | .973585 |
| | 80 | .983767 | .981984 | .980343 | .978320 | .976873 |
| | 90 | .985570 | .983983 | .982522 | .980722 | .979433 |
| | 100 | .987012 | .985583 | .984267 | .982645 | .981483 |

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VITA

David J. Lawton was born in Tampa, Florida on 27 December 1955. He graduated from Riverview High School, Sarasota, Florida in 1973. In June of 1977, he received a Bachelor of Science in Management/Mathematics from the United States Air Force Academy and was commissioned in the USAF. He completed Undergraduate Pilot Training at Williams AFB, Arizona and was awarded his wings in August 1978. He then served as a KC-135A pilot in the 42nd Aerial Refueling Squadron, Loring AFB, Maine until entering the School of Engineering, Air Force Institute of Technology in May 1983.

Permanent Address: 3873 Hawkeye Circle

Sarasota, Florida 33582

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This thesis uses three criteria to test for equality of p populations with underlying two parameter exponential distributions ($\theta$=location parameter, $\sigma$=scale parameter). These criteria use n random samples drawn from each of the p populations. The three criteria are based on the following three hypotheses:

1. $H_0 : \theta_1 = \theta_2 = \ldots = \theta_p,$
   $\sigma_1 = \sigma_2 = \ldots = \sigma_p$

2. $H_1 : \sigma_1 = \sigma_2 = \ldots = \sigma_p$
   $\theta_i$'s are unspecified $(i=1,2,\ldots,p)$

3. $H_2 : \theta_1 = \theta_2 = \ldots = \theta_p$
   given that $\sigma_1 = \sigma_2 = \ldots = \sigma_p$

against the general alternatives.

The asymptotic expansions of the distributions for $\lambda_1$, $\lambda_2$, and $\lambda_2$ are found based on the Neyman-Pearson likelihood ratio, where $\lambda_0$, $\lambda_1$, and $\lambda_2$ are the criteria for $H_0$, $H_1$, and $H_2$ respectively. The asymptotic expansions are computed using Bernoulli polynomials and a recursive relationship developed by Kalinin and Shalaevskii. Nine tables of percentage points are computed for each test statistic from the expansions where $p = 2(1)10$, $n = 10(1)20(5)50(10)100$, and $\alpha = .100$, .050, .025, .010, .005. These tables along with a practical illustration give the analyst a good technique that can be applied to many exponentially related situations.

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